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A High Accuracy Numerical Method and Error Analysis for Fourth Order Elliptic Eigenvalue Problems in Circular Domain

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Abstract. In this paper, an efficient spectral method is applied to solve fourth order elliptic eigenvalue problems in circular domain. Firstly, we derive the essential pole conditions and the equivalent dimension reduction schemes of the original problem. Then according to the pole conditions, we define the corresponding weighted Sobolev spaces. Together with the minimax principle and approximation properties of orthogonal polynomials, the error estimates of approximate eigenvalues are proved. Thirdly, we construct an appropriate set of base functions contained in approximation spaces and establish the matrix formulations for the discrete variational form, whose mass matrix and stiff matrix are all sparse so that we can solve the numerical solutions efficiently. Finally, we provide some numerical experiments to validate the theoretical results and algorithms.

AMS subject classifications: 65N30, 65N35

Key words: Fourth order elliptic eigenvalue problems, dimension reduction scheme, error analysis, numerical algorithms, circular domain.

1 Introduction

Fourth order elliptic eigenvalue problems arise in mechanics (vibration and buckling of plates [1–4]), inverse scattering theory [5] and the analysis of the stability of stationary solutions of the Navier-Stokes equations [6]. As a model, we consider the following fourth order elliptic eigenvalue problems:

$$\Delta^2 u - \alpha\Delta u + \beta u = \lambda u \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = \frac{\partial u}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega, \tag{1.1b}$$

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where α, β are nonnegative constants and $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega$.

There are various numerical methods for solving the fourth order elliptic eigenvalue problems, such as finite element methods, triangle element methods and so on. For the finite element methods, we can refer to [7–10]. For the triangle element methods, there has been a number of works that address spectral methods in triangular domains, such as the approximation methods by polynomials in triangle through mapping (see, e.g., [11–15]), using special nodal points (see, e.g., [16–19]) and non-polynomial functions in triangle (see, e.g., [20–24]). However, for the conforming finite element method, it requires globally continuously differentiable finite element spaces, which are difficult to construct and implement (in particular for the domain of polar geometries). For the nonconforming finite element method, a disadvantage is that such elements do not come in a natural hierarchy and existing nonconforming elements only involve low-order polynomials that are not efficient for capturing smooth solutions.

In a real application, we usually need to solve fourth order elliptic eigenvalue problems in some special domains as well. To the best of our knowledge, there has few report on spectral-Galerkin approximation for the fourth order elliptic eigenvalue problems in circular domain. The primary reason is that the polar transformation introduces singularity at the pole and the variable coefficients of the form $r^{\pm k}$ in polar coordinates [25], which involves some difficulties in developing efficient spectral-Galerkin approximation and corresponding error analysis. Thus, we will present in this paper an efficient spectral method to solve fourth order elliptic eigenvalue problems in circular domain. Firstly, we derive the essential pole conditions and the equivalent dimension reduction schemes of the original problem. Then according to the pole conditions, we define the corresponding weighted Sobolev spaces. Together with the minimax principle and approximation properties of orthogonal polynomials, the error estimates of approximate eigenvalues are proved. Thirdly, we construct an appropriate set of base functions contained in approximation spaces and establish the matrix formulations for the discrete variational form, whose mass matrix and stiff matrix are all sparse so that we can solve the numerical solutions efficiently. Finally, we provide some numerical experiments to validate the theoretical results and algorithms. Our method is mainly based on spectral-Galerkin approximation and some spectral collocation methods can also be considered [26, 27].

The rest of this paper is organized as follows. In Section 2, the dimension reduction scheme of the fourth order elliptic eigenvalue problems is presented. In Section 3, we derive the weak formulation and discrete formulation. In Section 4, we prove the error estimation of the approximation eigenvalues. In Section 5, we describe the details for an efficient implementation of the algorithm. In Section 6, we present several numerical experiments to demonstrate the accuracy and efficiency of our algorithm. Finally, in Section 7 we give some concluding remarks.