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A Compact Difference Scheme for the Time-Fractional Partial Integro-Differential Equation with a Weakly Singular Kernel

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Abstract. In this paper, we construct a compact difference scheme for the time-fractional partial integro-differential equation. This model involves two nonlocal terms in time, i.e., a Caputo time-fractional derivative and an integral term with memory. We obtain the stability and the discrete L_2 convergence with second-order in time and fourth-order in space by the energy method. Two numerical examples are provided to confirm the theoretical results.

AMS subject classifications: 45K05, 65M06, 35R11

Key words: Weakly singular kernel, compact difference scheme, time-fractional partial integro-differential equation, stability, convergence.

1 Introduction

Recently, fractional partial differential equations (FPDEs) have been widely used to describe phenomena in many scientific fields, such as control theory [18], universal response [17], financial mathematics, viscoelastic materials [1, 9, 21]. The analytical solutions of most FPDEs cannot be obtained, therefore many researchers [10, 12, 14, 15, 22–24] have concentrated on the numerical approaches for solving this kind of equation.

In this paper, we consider a compact difference scheme for the time-fractional partial integro-differential equation

$$ {}_0^C D_t^\alpha u(x,t) = \int_0^t (t-s)^{\beta-1} u_{xx}(x,s)ds + f(x,t), \quad x\in(0,L), \quad t\in(0,T], \tag{1.1}$$

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with the boundary conditions

$$u(0,t)=u(L,t)=0, \quad t\in(0,T], \tag{1.2}$$

and the initial condition

$$u(x,0)=\phi(x), \qquad x\in[0,L], \tag{1.3}$$

where $\alpha,\beta\in(0,1)$, L, T are positive constants, $f(x,t)$ and $\phi(x)$ are given functions, and ${}_0^CD_t^\alpha$ denotes the Caputo time-fractional derivative

$${}_0^CD_t^\alpha u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(x,s)}{\partial s}\frac{ds}{(t-s)^\alpha}.$$

Here, $\Gamma(\cdot)$ is the Gamma function.

Equation similar to (1.1) can be obtained from the standard partial integro-differential equation, i.e.,

$$u_t(x,t)=\int_0^t(t-s)^{\beta-1}u_{xx}(x,s)ds+f(x,t), \qquad t>0, \tag{1.4}$$

by replacing $u_t(x,t)$ by ${}_0^CD_t^\alpha u(x,t)$. We note that when α tends to 1, ${}_0^CD_t^\alpha u(x,t)$ converges to $u_t(x,t)$. Having the Caputo derivative in the generalized equation (1.1) can improve the modeling accuracy for describing physical phenomena involving viscoelastic forces. This equation often occurs in applications, such as linear viscoelastic mechanics [3, 20], heat conduction in materials with memory [6].

Different numerical methods for solving (1.4) have been developed. López-Marcos [12] investigated a backward-Euler scheme. Kim and Choi [8] considered spectral methods. Yan and Fairweather [26] presented orthogonal spline methods. Mclean and Thomée [15] derived the finite element method. Tang [23] analyzed a Crank-Nicolson scheme. Luo et al. [13] gave a compact difference scheme. Second-order backward differentiation formula methods were constructed in [2, 25] and alternating direction implicit schemes for high dimensional problems were provided in [7, 11, 19]. Replacing first term in (1.4) by Riemann-Liouville derivative and fixing $\beta=\frac{1}{2}$, a compact difference scheme with first-order in time was considered in [16].

So far, a high order numerical scheme for (1.1)-(1.3) has not been given. The main purpose of this work is to present a compact difference scheme with second-order in time and fourth-order in space for the problem (1.1)-(1.3). The L1-2 formula [5, 14] is used to discretize the Caputo time-fractional derivative. And we utilize a second-order method suggested by Diethelm et al. [4] to approximate the integral term. For the spatial directional derivative, a fourth-order compact difference method is employed.

The remainder of this paper is organized as follows. In Section 2, we propose a numerical scheme for the problem (1.1)-(1.3) and discuss the unique solvability. The stability and convergence are analyzed in Section 3. Two numerical examples are provided in Section 4 to evaluate the performance of the compact difference scheme. Finally, a conclusion is given for this work in Section 5.