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# On the Degenerate Scale of an Infinite Plane Containing Two Unequal Circles

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**Abstract.** In this paper, we analytically derived the degenerate scale of an infinite plane containing two unequal circles and numerically implemented by using the BEM. We provide two methods to analytically derive the degenerate scale. One is using the degenerate kernel and the other is using the conformal mapping. The closed-form fundamental solution of the two-dimensional Laplace equation is expanded to the degenerate kernel form in order to analytically study the degenerate scale in the BIE. Moreover, we used the technique of the conformal mapping in order to analytically study the degenerate scale in the complex variables. Then, a boundary value problem can be transformed into a Green's function. Finally, we prove the equivalence of the two analytical formulas derived by using the degenerate kernel and by using the complex variables. They are also examined by using the BEM. Good agreement is made. Besides, the case of the two equal circles is just a special one of the present formula.

**AMS subject classifications**: 45A05, 30B10

**Key words**: Degenerate scale, degenerate kernel, complex variables, bipolar coordinates, conformal mapping.

## 1 Introduction

Solving two-dimensional Laplace problems by using the boundary element method

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(BEM) or the boundary integral equation method (BIEM) is accurate and feasible. Nevertheless, it may lead to a non-uniqueness solution at the specific size for the two-dimensional Dirichlet problem. This critical size is called a degenerate scale. The degenerate scale is related to the Gamma contour [1], logarithmic capacity [2], transfinite boundary [3] and critical value [4]. We had already analytically studied the degenerate scale includes circle [5], ellipse [6, 7], regular $N$-gons [8] and infinite domain containing two equal circular cavities [9]. Therefore, it is important to predict where the degenerate scale may appear in the BIEM/BEM. Dijkstra [10] used the BEM to investigate the value of the degenerate scale for some geometry shapes [8]. Fig. 1(a) shows the degenerate scale (solid line) and the ordinary scale (dotted line) for two unequal circles. Fig. 1(b) indicates the drop of the minimum singular value due to the rank deficiency of the influence matrix in the BEM. Mathematically speaking, there are two ways to understand the degenerate scale. One is the unit logarithmic capacity corresponding to the conformal radius in the complex analysis [11]. The other is the non-uniqueness solution in the BIEM/BEM. Kimura [12] established a consistent theory which connects the estimation of the condition numbers and the convergence of numerical solutions. The definition of the logarithmic capacity was given in [13]. In recent decades, not only engineers but also applied mathematicians had interest in the topic [14, 15]. The linkage between the degenerate scale and the logarithmic capacity was discussed in [16–18]. Lin and Wu [19] revisited the degenerate scale problem by using the dimensional analysis. Chen [20] provided a numerical solution in antiplane elasticity by using the null-field boundary integral equation (BIE) for the degenerate scale.

For two-dimensional problems, the theory of complex variables is an analytical and popular tool. Rumely [2] mentioned the logarithmic capacity of many shapes using the conformal mapping in 1989. Chao et al. [21] derived the solution in the compact complex form by employing the analytical continuation method and the conformal mapping. Kuo et al. [8] investigated the regular $N$-gon domains by using the Riemann conformal mapping (Schwarz–Christoffel transformation). The results of the method and the numerical results agreed very well. In 2013, Kuo et al. [22] connected the unit logarithmic capacity and the degenerate scale by adopting the Riemann conformal mapping.

Nevertheless, the degenerate scale for the exterior problem is not popularly discussed as the interior problem. Rumely obtained the unit logarithmic capacity for an infinite plane with two identical circular holes by using elliptic functions [2]. Chen et al. [23, 24] investigated the numerical solution for the degenerate scale problem of the exterior multiply-connected region. In 2014, Corfdir and Bonnet [25] discussed the degenerate scale for a half-plane domain in the Laplace problem. In their research, they indicated that degenerate scale was associated with the type of the boundary condition on the line bounding the half-plane. Later, Chen [26] revisited the same problem by employing a null-field BIEM. Numerical results [26] also supported the finding in [25]. In those two papers [25, 26], they both constructed the corresponding Green's function by employing the image method. In 2017, Chen et al. [9] used the degenerate kernel of bipolar coordinates to find the analytical formula of the degenerate scale of the infinite