Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **12**, No. 6, pp. 1416-1437

DOI: 10.4208/aamm.OA-2019-0220
December 2020

A High-Order Well-Balanced Discontinuous Galerkin Method Based on the Hydrostatic Reconstruction for the Ripa Model

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Received 21 July 2019; Accepted (in revised version) 7 January 2020

Abstract. In this work, we present a high-order discontinuous Galerkin method for the shallow water equations incorporating horizontal temperature gradients (also known as the Ripa model), which exactly maintains the lake at rest steady state. Herein, we propose original numerical fluxes defined on the basis of the hydrostatic reconstruction idea and a simple source term approximation. This novel approach allows us to achieve the well-balancing of the discontinuous Galerkin method without complication. Moreover, the proposed method retains genuinely high-order accuracy for smooth solutions and it shows good resolution for discontinuous solutions at the same time. Rigorous numerical analysis as well as extensive numerical results all verify the good performances of the proposed method.

AMS subject classifications: 74S05

Key words: Ripa model, lake at rest steady state, source term, discontinuous Galerkin method, well-balancing property, hydrostatic reconstruction.

1 Introduction

Numerical simulation by high-order methods has a wide range of applications in environmental hydrodynamics [1,2]. Herein, we deal with the Ripa model, which is deduced from the shallow water equations (SWE) by incorporating the horizontal temperature gradients [3–5], often used to study ocean currents.

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The one-dimensional Ripa model has the following form:

$$\begin{cases} h_t+(hu)_x=0, \\ (hu)_t+\left(hu^2+\dfrac{1}{2}g\theta h^2\right)_x=-g\theta hB_x, \\ (h\theta)_t+(hu\theta)_x=0, \end{cases} \tag{1.1}$$

where x is the space, t is the time, h represents the water depth, u is the flow velocity, B is the bottom topography, and g is the gravitational constant. Correspondingly, $h+B$ stands for the free water surface level. Here, $\theta=\frac{\overline{\Theta}-\Theta}{\overline{\Theta}}$, with Θ and $\overline{\Theta}$ denoting the potential temperature and the reference potential temperature, accounts for the effects of the temperature on the fluid density. In addition, $P=\frac{1}{2}g\theta h^2$ represents the resultant of the pressure on each vertical divided by a reference fluid density. For the sake of easy presentation, we rewrite the system (1.1) in a compact vector form

$$U_t+F(U)_x=S(B,U), \tag{1.2}$$

with

$$U=(h,hu,h\theta)^{\mathrm{T}}, \quad F(U)=\left(hu,hu^2+\frac{1}{2}g\theta h^2,hu\theta\right)^{\mathrm{T}} \quad \text{and} \quad S(B,U)=(0,-g\theta hB_x,0)^{\mathrm{T}}$$

being the vector of the conservative variables, the physical flux, as well as the vector of the source terms, respectively.

From the mathematical point of view, the system (1.1) combines hyperbolic balance laws, which are associated to particular steady states when $U_t=0$. Especially, the one-dimensional Ripa model (1.1) admits the following two types of lake at rest steady states [6]:

$$u=0, \quad \theta=\text{constant}, \quad h+B=\text{constant}, \tag{1.3a}$$

$$u=0, \quad B=\text{constant}, \quad P=\frac{1}{2}g\theta h^2=\text{constant}. \tag{1.3b}$$

Under the steady state (1.3a), the flux gradient is non-zero and is exactly balanced by the source term (i.e., $F(U)_x=S(B,U)$).

To reproduce this asymptotic behavior of the system (1.1), the well-balancing property of a numerical method, which is the property to maintain the exact balance between the flux gradient and the source term at the discrete level accurately preserving the steady state up to the machine accuracy, is welcome. A numerical method satisfying the well-balancing property is named well-balanced method. Moreover, it is important to note that, compared with the non well-balanced counterparts, the well-balanced methods can accurately resolve small perturbations of the steady state on relatively coarse meshes [7–9], saving computational efforts considerably.