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A Parallel Pressure Projection Stabilized Finite Element Method for Stokes Equation with Nonlinear Slip Boundary Conditions

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Abstract. For the low-order finite element pair P_1-P_1, based on full domain partition technique, a parallel pressure projection stabilized finite element algorithm for the Stokes equation with nonlinear slip boundary conditions is designed and analyzed. From the definition of the subdifferential, the variational formulation of this equation is the variational inequality problem of the second kind. Each subproblem is a global problem on the composite grid, which is easy to program and implement. The optimal error estimates of the approximate solutions are obtained by theoretical analysis since the appropriate stabilization parameter is chosen. Finally, some numerical results are given to demonstrate the hight efficiency of the parallel stabilized finite element algorithm.

AMS subject classifications: 68W10, 76M10, 76D07

Key words: Stokes equations, nonlinear slip boundary conditions, pressure projection, full domain partition, parallel stabilized finite element algorithm.

1 Introduction

In this paper, we shall study the incompressible Stokes problem:

$$\begin{cases} -\nu\Delta u+\nabla p=f & \text{in } \Omega, \\ \nabla\cdot u=0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

with the following homogeneous Dirichlet boundary condition on Γ and nonlinear slip boundary conditions on S:

$$\begin{cases} u=0 & \text{on } \Gamma, \\ u_n=0, \quad -\sigma_\tau(u)\in g\partial|u_\tau| & \text{on } S, \end{cases} \tag{1.2}$$

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where $\Omega\subset\mathbb{R}^2$ is a bounded convex domain with a Lipschitz-continuous boundary $\partial\Omega$. $\Gamma\cap S=\emptyset$, $\overline{\Gamma\cup S}=\partial\Omega$. $g\geq 0$ is a scalar function stands for the barrier or threshold slip function; $u_n=u\cdot n$ and $u_\tau=u\cdot\tau$ are the normal and tangential components of the velocity on S respectively; the Cauchy stress vector σ is defined as $\sigma_i=\sigma_i(u,p)=(\nu e_{ij}(u)-p\delta_{ij})n_j$, here $e_{ij}(u)=\frac{\partial u_i}{\partial x^j}+\frac{\partial u_j}{\partial x^i}$, $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$, write $\sigma_\tau(u)=\sigma-\sigma_n n$ is the tangential component of σ. Set $\phi:\mathbb{R}\to\overline{\mathbb{R}}=(-\infty,+\infty]$ and it contains the properties of convexity and weak semi-continuity from below (ϕ is not identical with $+\infty$). The subdifferential set $\partial\phi(a)$ denotes a subdifferential of the function ϕ at the point a is defined by: $\partial\phi(a)=\{b\in\mathbb{R}:\phi(h)-\phi(a)\geq b(h-a),\ \forall h\in\mathbb{R}\}$.

As a fundamental equations of fluid mechanics, Stokes equations are widely used, and relevant mathematical research can be found in [1,2]. In some fluid problems, such as blood flow in a vein of an arterial sclerosis patient, the avalanches in water and rocks, flow through a canal or a drainpipe with its bottom covered by sherbet of mud and pebbles, oil flows above or below the sand, the traditional Dirichlet boundary conditions are no longer applicable, so there is practical application value in studying Stokes equations with nonlinear slip boundary conditions. The boundary condition (1.2) was introduced by Fujita in [3], subsequently, there have been many studies about the properties of the solution (cf. [3–9]), including existence, uniqueness, regularity, and continuous dependence on data for the Stokes and Navier-Stokes equations with boundary conditions (1.2). Also, there are a lot of finite element methods for Stokes and Navier-Stokes equations which satisfying inf-sup condition, like the penalty finite element methods in [10–12], discontinuous Galerkin methods in [13] and so on.

The low-order finite element pair P_1-P_1 does not satisfy the inf-sup condition, however, it has been widely used in practical calculations due to its simple and regular data structure, which makes calculations small and accurate. In order to solve the pressure oscillation caused by low-order finite element pair P_1-P_1, many stabilization techniques are used to compensate for the inf-sup condition, for example, two-level penalized finite element methods [14], regularization method [15], local Gauss integrations method [16], variational multiscale method [17] and pressure projection stabilization method [18]. The pressure projection stabilization method proposed in this paper is based on the literature [19]. Compared with other stabilization methods, this method does not need to calculate high-order derivatives or boundary integrals, and there is no grid nesting.

On the basis of literature [20,21], we propose a more efficient algorithm for equations (1.1)-(1.2), which is the parallel algorithm. Then, by choosing the appropriate stabilization parameters, we can obtain the optimal error estimates for approximate solutions. The idea of local and parallel finite element computations was from Xu's work [22], which have become a hot research topic with the advent of parallel computers. However, there is little research on the local and parallel computations for the Stokes or Navier-Stokes equations with nonlinear slip boundary conditions. A finite element solution of some partial differential equations usually consists of low-frequency components and high frequency components. The low-frequency components describe a wide range of