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Finite Element Methods for Nonlinear Backward Stochastic Partial Differential Equations and Their Error Estimates

Xu Yang[1] and Weidong Zhao[2,*]

[1] *School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221116, China*
[2] *School of Mathematics, Shandong University, Jinan, Shandong 250100, China*

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Abstract. In this paper, we consider numerical approximation of a class of nonlinear backward stochastic partial differential equations (BSPDEs). By using finite element methods in the physical space domain and the Euler method in the time domain, we propose a spatial finite element semi-discrete scheme and a spatio-temporal full discrete scheme for solving the BSPDEs. Errors of the schemes are rigorously analyzed and theoretical error estimates with convergence rates are obtained.

AMS subject classifications: 60H15, 60H35, 65C30, 65M60

Key words: Backward stochastic partial differential equations, finite element method, error estimate.

1 Introduction

Consider the following backward stochastic partial differential equations (BSPDEs):

$$-du(t,x)-\mathcal{L}u(t,x)dt=f\big(t,\nabla u(t,x),u(t,x),v(t,x)\big)dt-v(t,x)dW(t), \tag{1.1}$$

for $t\in[0,T)$, $x\in D$. The associated boundary and terminal conditions are given by

$$\begin{cases} u(t,x)=v(t,x)=0, & x\in\partial D,\\ u(T,x)=u_T(x), & x\in D. \end{cases}$$

Here $T>0$ is a positive constant and $D\subset\mathbb{R}^d$, $(d=1,2,3)$ is a bounded convex domain with a sufficiently smooth boundary ∂D. The operator $\mathcal{L}$ is a second order elliptic operator

*Corresponding author.
Emails: xuyang96@cumt.edu.cn (X. Yang), wdzhao@sdu.edu.cn (W. Zhao)

defined by

$$\mathcal{L}u=:\sum_{j,k=1}^{d}\frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u}{\partial x_k}\right)-a_0(x)u.$$

Moreover, $W=\{W(t)\colon t\in[0,T]\}$ is a standard Wiener process defined on a completed probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ with $\mathbb{F}=:\{\mathcal{F}_t\colon t\in[0,T]\}$ being the augmented natural filtration generated by W. Also, u_T is an $\mathcal{F}_T$-measurable random field. In fact, the BSPDEs (1.1) can be mathematically interpreted as the following integral form

$$\begin{aligned}&u(t,x)-\int_t^T\mathcal{L}u(s,x)ds\\=&u_T(x)+\int_t^T f(s,\nabla u(s,x),u(s,x),v(s,x))ds-\int_t^T v(s,x)dW(s).\end{aligned}\tag{1.2}$$

Precise assumptions on the operator $\mathcal{L}$, the function f, and the terminal function u_T will be discussed in Section 2.

The above BSPDEs play an essential role in many real applications. In particular, it serves as the adjoint equations in stochastic optimal control problems governed by stochastic parabolic equations [3, 11, 34]. Also, the nonlinear BSPDEs appears as the value functions in the optimal control problems of non-Markovian SDEs [20]. Other applications of BSPDEs include the nonlinear filtering problems [18], and mathematical finance [9, 16]. Recently, great attention has been paid to develop theoretical analysis of BSPDEs. Hu and Peng [11, 12] were the first to investigate the well-posedness of adapted solutions to semilinear backward stochastic evolution equations. The existence and uniqueness, as well as the regularity, of the adapted solutions for a class of BSPDEs was discussed in [16, 24, 33]. For more recent developments, one can refer to [1, 7, 9, 22] and references therein.

As analytic solutions to BSPDEs are seldom available, numerical methods become popular approaches for solving BSPDEs. As a generalization of backward stochastic differential equations (BSDEs), the associated numerical methods for BSPDEs have not been well studied. In contract, the numerical methods for BSDEs have been well developed in recent years [2, 4, 10, 15, 21, 23, 27, 28, 30, 32]. Up to now, there exist only a very limited number of works [8, 26] devoted to this field. In [8], finite element methods for linear forward-backward stochastic heat equations were considered and a rigorous convergence analysis for the spacial semi-discrete scheme was presented for linear BSPDEs. A semidiscrete Galerkin scheme based on spectral method for BSPDEs was studied in [26].

The primary goal of this paper is to develop and analyze numerical schemes which are used to approximate the solutions of nonlinear BSPDEs (1.1). More precisely, we study the strong approximation errors caused by spatial semi-discretization and space-time full-discretization of (1.1). We first consider semidiscrete finite element method for (1.1) and by a combination of the finite element method together with a linear implicit Euler time-stepping scheme, we also investigate a spatio-temporal discretization of (1.1). For both cases, we get the error estimates with precise strong convergence rate shown in