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The Crank-Nicolson/Explicit Scheme for the Natural Convection Equations with Nonsmooth Initial Data

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Abstract. In this article, a Crank-Nicolson/Explicit scheme is designed and analyzed for the time-dependent natural convection problem with nonsmooth initial data. The Galerkin finite element method (FEM) with stable MINI element is used for the velocity and pressure and linear polynomial for the temperature. The time discretization is based on the Crank-Nicolson scheme. In order to simplify the computations, the nonlinear terms are treated by the explicit scheme. The advantages of our numerical scheme can be list as follows: (1) The original problem is split into two linear subproblems, these subproblems can be solved in each time level in parallel and the computational sizes are smaller than the origin one. (2) A constant coefficient linear discrete algebraic system is obtained in each subproblem and the computation becomes easy. The main contributions of this work are the stability and convergence results of numerical solutions with nonsmooth initial data. Finally, some numerical results are presented to verify the established theoretical results and show the performances of the developed numerical scheme.

AMS subject classifications: 65M10, 65N30, 76Q10

Key words: Natural-Convection equations, Crank-Nicolson/Explicit scheme, nonsmooth initial data, error estimates.

1 Introduction

In this paper, we consider the following natural convection equations:

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$$
\begin{cases}
\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = -\kappa\nu^2 jT + \mathbf{f} & \text{in } \Omega\times(0,T_{time}^{final}], \\
\text{div }\mathbf{u} = 0 & \text{in } \Omega\times(0,T_{time}^{final}], \\
T_t - Pr^{-1}\nu\Delta T + \mathbf{u}\cdot\nabla T = g & \text{in } \Omega\times(0,T_{time}^{final}], \\
\mathbf{u} = 0, \quad T = 0 & \text{on } \partial\Omega\times(0,T_{time}^{final}], \\
\mathbf{u}(x,0) = \mathbf{u}_0, \quad T(x,0) = T_0 & \text{on } \Omega\times\{0\},
\end{cases}
\tag{1.1}
$$

where $\mathbf{u}$, T, p are the velocity, temperature and the pressure, $\mathbf{f}$ and g are the body forces, Ω is a bounded convex polygonal domain, the parameters ν, κ and Pr are the viscosity, Groshoff and Prandtl numbers, $j=(0,1)^T$ is the vector of gravitational acceleration, $T_{time}^{final}>0$ is the final time.

The natural convection problem is an important system with dissipative nonlinear terms in atmospheric dynamics (see [4, 25]), it not only inherits all difficulties of the Navier-Stokes equations, but also contains strong coupling among variables and nonlinear terms. Hence, finding the numerical solutions becomes a difficult task, and several efficient numerical methods have been developed in recent years, for examples, [5, 8, 27] for the discontinuous methods, [24, 26] for the lattice Boltzmann method, [7] for the stabilized method, [21, 23] for the iterative schemes.

As a classical second order scheme, the Crank-Nicolson scheme has been used to treat various problems. Here we just refer to [20, 22] for the linear problems, [3] for the semilinear parabolic problem and the reference therein as the examples. Generally speaking, the implicit scheme for nonlinear term is unconditionally stable and has optimal error estimates, but we need to treat a nonlinear problem at each step and a lot of computing cost is required. In order to simplify the computations, some variants of the Crank-Nicolson scheme were developed, for examples, the Crank-Nicolson extrapolation scheme [12, 14, 28], the Crank-Nicolson/Newton scheme [9]. The explicit scheme for nonlinear term is another way to treat the nonlinear term, one of the most important advantages is that the discrete algebraic system with a constant coefficient matrix is obtained at each time level. However, a restriction on the time-step was required. For examples we can refer to the Crank-Nicolson/Adams-Bashforth scheme [10, 14, 16, 31] and the references therein.

In this paper, we consider the Crank-Nicolson/Explicit scheme for the natural convection equations with nonsmooth initial data. In this way, the origin problem is split into two linear subproblems, and these subproblems with the constant coefficient matrix can be solved easily in each time level. Compared with [32, 33], the main contributions can be list as follows:

(1) Under some restrictions on time step, almost unconditional stability results of numerical solutions in various norms are established with nonsmooth initial data.

(2) By introducing the weight function and using the negative norm technique, under the same time step conditions, we obtain that the Crank-Nicolson/Explicit