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A New Mixed Finite Element Method for Biot Consolidation Equations

Luoping Chen[1,3] and Yan Yang[2]*

[1] *School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan 611756, China*
[2] *School of Sciences, Southwest Petroleum University, Chengdu, Sichuan 610500, China*
[3] *MOE Key Lab for Neuro Information, School of Life Science and Technology, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, China*

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Abstract. In this paper, we study a new finite element method for poroelasticity problem with homogeneous boundary conditions. The finite element discretization method is based on a three-variable weak form with mixed finite element for the linear elasticity, i.e., the stress tensor, displacement and pressure are unknown variables in the weak form. For the linear elasticity formula, we use a conforming finite element proposed in [11] for the mixed form of the linear elasticity and piecewise continuous finite element for the pressure of the fluid flow. We will show that the newly proposed finite element method maintains optimal convergence order.

AMS subject classifications: 65M10, 78A48

Key words: Biot consolidation equations, linear elasticity, finite element method, convergence analysis.

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1 Introduction

General theory describing the consolidation of a porous elastic soil is very important in application, for example, predicting the behavior of foundation resting on a saturated clay is an important problem in foundation engineering. The foundation allows for the occurrence of finite geometry changes and finite elastic strains during the consolidation process. This theory of poroelasticity addresses the time-dependent coupled process between the deformation of porous materials and the fluid flow inside. The governing equations have been cast in a rate form and laws which determine deformation and pore

*Corresponding author.
Emails: cherrychen@home.swjtu.edu.cn (L. P. Chen), yyan2011@163.com (Y. Yang)

fluid flow are Hookes's law and Darcy's law. The theoretical basis of consolidation was established by Terzaghi [27], then, Biot generalized the theory to three dimensional transient consolidation [5, 6]. Since then, poroelastic theory has been used in a diverse range of science and engineering application, for instance, CO_2 sequestration in environmental engineering [15, 16] are important applications of poroelasticity. Recently, research in poroelasticity has been a surge in activity, not only because of the application described above, but also due to emerging applications in biomechanics engineering such as biological soft tissue modeling including arterial walls, skin, cardiac muscle and articular cartilage [13, 17, 25, 28].

There is an extensive literature on numerical methods for poroelasticity. The most commonly used numerical discretization are based on the two-fields model problem, i.e., displacement $\boldsymbol{u}$ and fluid pressure p as its unknown variables. Standard centered finite difference methods are studied for both one and two dimensional of the model problem [8, 9]. Also, continuous finite element methods, such as Taylor Hood element and Mini element are employed for the displacement and pressure, see [14, 18] as examples. However, it is well known that the approximation by centered difference method and some Galerkin finite element method often exhibit nonphysical oscillation in the pressure of the fluid flow. Therefore, for the two-field model problem, finite difference based on staggered grids [9], cell-centered finite volumn discretization [3, 19] are studied and proved to be the stable discretization. For the two-field model problem with classical stable finite element methods (such as Mini, stabilized $P_1 - P_1$ element etc.), the approximate pressure variable still appears nonphysical oscillation under the condition of low permeability or small time step size. Authors in [23] claim that the well known inf-sup stable pair spaces does not necessary provide oscillation-free solution and stabilized term is added to guarantee the monotonicity of the solution to eliminate oscillation. Finite element methods based on classical three-fields model problem, i.e., displacement, fluid flux, pore pressure, are also studied trying to solve the nonphysical oscillations of the fluid pressure. Based on the analysis of locking reason in pure linear elasticity problem, Phillips and Wheeler make arguments that under certain conditions, nonphysical oscillation of fluid pressure may be produced by locking in the solid elasticity. Therefore, nonconforming finite element methods with couple continuous and discontinuous Galerkin (DG) methods for the displacement and a mixed finite element method for the flow variables are investigated, see [20–22]. Stabilized term with face bubble is added in [24] for the finite element (with linear P_1 element for $\boldsymbol{u}$ and lowest Raviart Thomas element for fluid flux and pressure) to guarantee uniform error bounds. Other three-field based finite element methods are also studied, we refer [26, 30] as well as the references therein for further investigation. Four fields formulations for the poroelasticity problem are also investigated. Least squares mixed finite element methods for the stress tensor/displacement/fluid velocity/fluid pressure four-field formulation have been proposed by Korsawe and Starke [12] and Tchonkova et al. [26]. Yi [29] developed a four-field discretization method with displacement, stress tensor in solid subproblem satisfying Hellinger-Reissner variational principle (using Arnold-Winther element)