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A Conservative Gradient Discretization Method for Parabolic Equations

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Abstract. In this paper, we propose a new conservative gradient discretization method (GDM) for one-dimensional parabolic partial differential equations (PDEs). We use the implicit Euler method for the temporal discretization and conservative gradient discretization method for spatial discretization. The method is based on a new cell-centered meshes, and it is locally conservative. It has smaller truncation error than the classical finite volume method on uniform meshes. We use the framework of the gradient discretization method to analyze the stability and convergence. The numerical experiments show that the new method has second-order convergence. Moreover, it is more accurate than the classical finite volume method in flux error, L^2 error and L^∞ error.

AMS subject classifications: 65M08, 35K10

Key words: Gradient discretization method, mass conservation, parabolic equations.

1 Introduction

Parabolic equations are a typical type of time-dependent problems. Many time-dependent physical processes, such as heat conduction problems, underground engineering, oil recovery, and nuclear waste disposal, image analysis, can be described by parabolic equations [11, 12, 15, 22]. Many numerical methods have been applied to parabolic equations, such as finite difference methods [2, 32], finite element methods [17, 31, 37, 40, 41], discontinuous Galerkin methods [6, 29, 38], spectral Galerkin method [36], weak Galerkin method [39], and so on.

However, conventional numerical methods usually do not have local mass conservation property. Mass conservation property is crucial physically, and a numerical scheme

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may produce a non-physical solution without local mass conservation. There have been a lot of efforts dedicated to the research of mass conservation preserving schemes. Different types of finite volume schemes have been proposed for the elliptic and parabolic equations. An implicit multi-point flux approximation on quadrilateral meshes is discussed in [1]. In [28], a minimal stencil finite volume scheme is studied. A monotone finite volume method is introduced in [35]. For the conservative finite difference methods, they also have been applied to various PDEs, including semilinear parabolic equation [10], hyperbolic conservation laws [9, 18], Helmholtz problem [23], acoustic wave equations [24,25], the kinetic and fluid simulations [33], multi-component flow computations, and transport process [30]. In recent years, high order conservative discontinuous Galerkin methods are also proposed for many PDEs, such as Klein-Gordon-Schrödinger equations [4], nonlinear electromagnetic Schrödinger equations [34], radiative transfer equations [26], and hyperbolic conservative equations [5, 27].

The gradient discretization method (GDM) is an efficient numerical method for solving linear and nonlinear elliptic and parabolic partial differential equations, see [13]. The main idea of the GDM is to use discrete spaces and discrete differential operators to mimic the original continuous spaces and differential operators in a variational formulation. In fact, the gradient discretization method is a highly flexible framework consisting of a large family numerical methods, such as conforming finite element method, nonconforming finite element method, and two-point flux finite volume method. In recent years, many numerical methods are analyzed under the framework of GDM, including the vertex approximate gradient (VAG) methods [20, 21], multi-point flux approximation method [1], hybrid mimetic mixed methods [14, 15], nodal mimetic finite difference methods [3,14], discrete duality finite volume methods [7,8,16], and the nine-point stencil finite volume method [22].

In this paper, we propose a conservative gradient discretization method for 1-d parabolic equation, and analyze the stability and convergence of the scheme under the framework of GDM. The main advantages of this scheme are

- it has only cell-centered unknowns;
- the stencil only includes neighbor cells;
- it can be applied to general non-uniform meshes;
- it is locally conservative;
- the resulting linear system is symmetric and positive definite;
- the diffusion coefficient can be discontinuous or nonlinear;
- for linear cases, the convergence orders in H^1 and L^2 norms are derived; for quasilinear cases, the convergences of H^1 and L^2 norms are proved;