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A New Fifth-Order Finite Difference WENO Scheme for Dam-Break Simulations

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Abstract. In this paper, a fifth-order weighted essentially nonoscillatory scheme is presented for simulating dam-break flows in a finite difference framework. The new scheme is a convex combination of two quadratic polynomials with a fourth-degree polynomial in a classical WENO fashion. The distinguishing feature of the present method is that the same five-point information is used but smaller absolute truncation errors and the same accuracy order in the smooth region are obtained. The new nonlinear weights are presented by Taylor expansion of the smoothness indicators of the small stencils to sustain the optimal fifth-order accuracy. The linear advection equation, nonlinear scalar Burgers equation, and one- and two-dimensional Euler equations are used to validate the high-order accuracy and excellent resolution of the presented method. Finally, one- and two-dimensional Saint-Venant equations are tested by using the new fifth-order scheme to simulate a dam-break flow.

AMS subject classifications: 65M60, 35L65

Key words: WENO scheme, smoothness indicators, shallow water equation, hyperbolic conservation laws.

1 Introduction

Dam-break simulations are very important in hydraulic engineering, as the work is related to people's lives and property safety. The governing equation of a dam break is the shallow water equation, also referred to as the Saint-Venant system [1]. It is widely applied in ocean and hydraulic engineering. This system describes the flow as a conservation law with an additional source. In one-dimensional space, the equation takes the

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form [2]

$$\frac{\partial h}{\partial t}+\frac{\partial(hu)}{\partial x}=0, \tag{1.1a}$$

$$\frac{\partial(hu)}{\partial t}+\frac{\partial(hu^2+\frac{1}{2}gh^2)}{\partial x}=-gh\frac{\partial b}{\partial x}, \tag{1.1b}$$

where b is the vertical height of the bottom topography, h denotes the water height, u is the velocity of the fluid and g is the gravitational constant. Because of the practical importance of the shallow water equation, studies on numerical methods for this system have attracted much attention in the past few years [1–4] and recently [5–9]. In the homogeneous case, the system is equivalent to a Euler system, which satisfies hyperbolic conservation laws.

One characteristic of the hyperbolic conservation laws is that they may develop discontinuities in the solution even if the initial conditions are smooth. Thus, classical numerical methods that depend on Taylor expansion fail to obtain an approximate solution for hyperbolic conservation laws. In the past few decades, many high-order finite difference or finite volume methods have been investigated to solve hyperbolic conservation laws [10–17]. However, high-order approximation leads to spurious oscillations in the solution. To overcome this phenomenon, total variation diminishing (TVD) schemes constructed by Harten [10, 18] are based on the principle that the total variation in the approximation of the numerical solution must be nonincreasing in time, however, it has been found that TVD schemes are at most first-order accurate near smooth extrema [19]. To improve the accuracy of TVD schemes, essentially non-oscillatory (ENO) and weighted essentially non-oscillatory (WENO) schemes have been applied quite successfully to solve problems with strong shocks, contact discontinuities and sophisticated smooth structures [20–30]. A series of ENO schemes were developed by Harten et al. [12] to solve the one-dimensional problem, where instead of using a single fixed stencil to approximate spatial fluxes, the ENO scheme used a set of candidate stencils determined by smoothness indicators. However, the ENO scheme is not effective, as such adaption of stencils is not necessary in smooth regions. Thus, the WENO scheme was introduced by Liu et al. [20] to overcome the drawbacks of the ENO scheme while maintaining robustness and high-order accuracy, in such schemes, spatial derivatives are calculated by using a convex combination of numerical fluxes associated with each candidate stencil. Jiang and Shu developed a classical method to calculate the smoothness indicators of the stencils, called the WENO-JS scheme, where the convergence accuracy of the WENO-JS scheme is fifth order in theory, but its actual rate of convergence is less than fifth order at critical points for many problems. The mapped WENO scheme (WENO-M) [31] was developed to have a formal fifth-order convergence at critical points of a smooth solution. This scheme uses a mapping function $k(\varepsilon)$, which renders the nonlinear weights closer to optimality to satisfy sufficient criteria for fifth-order convergence, however, the CPU cost is 1.25 times that of the WENO-JS scheme. By a simple combination of classical smoothness indicators, the WENO-Z scheme was presented in [15, 32], which not only