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Modified Two-Grid Algorithm for Nonlinear Power-Law Conductivity in Maxwell's Problems with High Accuracy

Changhui Yao[1,2,*] and Yanfei Li[2]

[1] *School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, Henan 450001, China*
[2] *State Key Laboratory of Space Weather, Chinese Academy of Sciences, Beijing 100190, China*

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Abstract. In this paper, we develop the superconvergence analysis of two-grid algorithm by Crank-Nicolson finite element discrete scheme with the lowest Nédélec element for nonlinear power-law conductivity in Maxwell's problems. Our main contribution will have two parts. On the one hand, in order to overcome the difficulty of misconvergence of classical two-grid method by the lowest Nédélec element, we employ the Newton-type Taylor expansion at the superconvergent solutions for the nonlinear terms on coarse mesh, which is different from the numerical solution on the coarse mesh classically. On the other hand, we push the two-grid solution to high accuracy by the postprocessing interpolation technique. Such a design can improve the computational accuracy in space and decrease time consumption simultaneously. Based on this design, we can obtain the convergent rate $\mathcal{O}(\Delta t^2+h^2+H^{\frac{5}{2}})$ in three-dimension space, which means that the space mesh size satisfies $h=\mathcal{O}(H^{\frac{5}{4}})$. We also present two examples to verify our theorem.

AMS subject classifications: 65N30, 65N15

Key words: Maxwell's equation, two-grid algorithm, Nédélec element, postprocessing, superconvergence.

1 Introduction

Two-grid method is a numerical approximation based on two-grid of different sizes in order to improve computational efficiency [1, 2, 4, 28, 29]. Generally, it is used to solve

*Corresponding author.
Emails: chyao@lsec.cc.ac.cn (C. H. Yao), lyanfei@gs.zzu.edu.cn (Y. F. Li)

the problems with non-symmetry, indefiniteness, large wave number or nonlinearity behaving, etc. The basic idea is to calculate the source problem on the coarse mesh with grid size H and calculate linearized problem on the fine mesh with grid mesh $h<H$. The advantages of this approach over the general implicit format is that the solution obtained on the coarse mesh is taken as the prediction of that on the fine mesh, and then the numerical solution is obtained through the corresponding iterative algorithm, which can save the CPU times and improves workloads.

In numerical simulation of electromagnetic system by finite element methods, we employ Nèdèlec elements or edge elements to approach $H(curl,\Omega)$. In order to decrease the computational cost, two-grid method is one of choices. Unfortunately, Zhong's work cites a counter-example [3] in Maxwell's equation. The natural reason is that the approximation properties of Nédélec elements or edge elements are not as good as that of nodal elements in the L^2-norm. Therefore, few work can be found after that of Zhong's work in 2013. In 2018, the authors take the two-level or two-grid method as the postprocessing techniques to simulate Navier-Stokes equations [4]. In fact, the postprocessing with superconvergence has been proposed by Lin's group from 1990s [5–10, 12–15], where the authors construct higher postprocessing operators based on lower numerical solutions by finite element methods to improve the convergent order rate. In this paper, we use postprocessing technique with superconvergence to modify two-grid algorithm for electromagnetic system in order to ensure matched error estimate between coarse mesh and fine mesh for the first type Nédélec elements [16].

We consider nonlinear power-law conductivity in Maxwell's problems,

$$\epsilon\frac{\partial \mathbf{E}}{\partial t}-\nabla\times\mathbf{H}+\mathbf{J}(\mathbf{E})=\mathbf{F} \qquad \text{in } \Omega\times(0,T], \tag{1.1a}$$

$$\mu\frac{\partial \mathbf{H}}{\partial t}+\nabla\times\mathbf{E}=0 \qquad \text{in } \Omega\times(0,T], \tag{1.1b}$$

$$\mathbf{n}\times\mathbf{E}=0 \qquad \text{in } \partial\Omega\times(0,T], \tag{1.1c}$$

$$\mathbf{E}(0,\mathbf{x})=\mathbf{E}_0(\mathbf{x}), \quad \mathbf{H}(0,\mathbf{x})=\mathbf{H}_0(\mathbf{x}) \qquad \text{in } \Omega, \tag{1.1d}$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^3$ with a sufficiently smooth boundary $\partial\Omega$. $\mathbf{E}$, $\mathbf{H}$ denote electric and magnetic field respectively, $\mathbf{n}$ is outward unit normal vector. ϵ and μ are the electric permittivity and the magnetic permeability of the medium, respectively. We also assume that ϵ and μ are constants for convenience.

In [24], Yin proved existence of the weak solution for a nonlinear function $\mathbf{J}(\mathbf{E})=\sigma(x,|\mathbf{E}|)\mathbf{E}$, with $\sigma(x,|\mathbf{E}|)$ is a monotone function. This domain is occupied by a nonlinear conducting material with electric conductivity $\sigma(x,|\mathbf{E}|)$, which is assumed to be a monotone function of the form $|\mathbf{E}|^{\alpha-1}$, with $\alpha>0$ [17, 18]. For simplification, we set $\mathbf{J}(\mathbf{E})=|\mathbf{E}|^{\alpha-1}\mathbf{E}$.

The system arises in some physical application programs, for example, constitutive law for type-II superconductors [19, 20] and molding of the nonlinear conductivity of the charge-density wave state of $NbSe_3$ [21], microwave heating [22], and special application for electric conductivity switches on eddy current [23]. The numerical analysis of