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A Galerkin-Characteristic Finite Element Method for Three-Dimensional Convection-Dominated Problems

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Abstract. We present an efficient Galerkin-characteristic finite element method for the numerical solution of convection-diffusion problems in three space dimensions. The modified method of characteristics is used to discretize the convective term in a finite element framework. Different types of finite elements are implemented on three-dimensional unstructured meshes. To allocate the departure points we consider an efficient search-locate algorithm for three-dimensional domains. The crucial step of interpolation in the convection step is carried out using the basis functions of the tetrahedron element where the departure point is located. The resulting semi-discretized system is then solved using an implicit time-stepping scheme. The combined method is unconditionally stable such as no Courant-Friedrichs-Lewy condition is required for the selection of time steps in the simulations. The performance of the proposed Galerkin-characteristic finite element method is verified for the transport of a Gaussian sphere in a three-dimensional rotational flow. We also apply the method for simulation of a transport problem in a three-dimensional pipeline flow. In these test problems, the method demonstrates its ability to accurately capture the three-dimensional transport features.

AMS subject classifications: 65M60, 65N30, 76D33

Key words: Three-dimensional convection-diffusion equations, Galerkin-characteristic method, finite elements, unstructured grids, convection-dominated problems.

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1 Introduction

In many practical transport problems from engineering and mathematical sciences, the governing equations involve convection-dominated flow systems. This class of problems has important applications in a variety of physical and engineering areas such as weather prediction, ocean circulation, petroleum reservoir among others. The physical phenomena in these areas can be modeled by transport-diffusion equations with the property that the convective terms are distinctly more important than the diffusive terms; particularly when certain nondimensional parameters reach high values. Examples of these parameters include the Peclet number for convection-diffusion equations and the Reynolds number for incompressible Navier-Stokes equations. Furthermore, it is well known that for large values of these parameters, the convective terms are a source of computational difficulties and nonphysical oscillations. In addition, steep fronts and boundary layers are among the difficulties that most of Eulerian finite element methods fail to resolve accurately, see for example [12]. In general, the Eulerian methods use fixed grids and incorporate some upstream weighting in their formulations to stabilize the spatial discretization. As examples of Eulerian methods we mention the Petrov-Galerkin methods, the streamline diffusion methods, discontinuous Galerkin methods and also many other methods such as the high resolution methods from computational fluid dynamics, in particular, the Godunov methods and the essentially non-oscillatory methods, see [3, 19] among others. The main limitation of these methods is the stability conditions which impose a severe restriction on the size of time steps taken in the simulations. Needless to mention that the complexity and the huge memory requirements of such methods make the state of art in this area more advanced for two-dimensional problems than their three-dimensional counterparts.

The modified method of characteristics is second-order accurate in space and time provided the characteristic curves are exactly calculated, compare for example [14, 22]. However, for general convection problems, the accuracy of the method depends on the order of the interpolation procedure used to calculate the departure points and on the time integration procedure. Theoretical analysis of convergence and stability of the Galerkin-characteristic finite element method have been carried out in many studies, see for example [5, 6, 14, 22]. It should be stressed that research works presented in these references have focused on the analysis of the Galerkin-characteristic finite element method with no algorithmic formulation of the method for engineering applications. Furthermore, our present study differs from the investigations reported in [5, 6, 14, 22] in the fact that it focuses on the computational implementation of the method for three-dimensional convection-diffusion equations on unstructured meshes and it also presents a comprehensive numerical assessment of the method using several test problems. The main objective of our work is the development of a highly efficient Galerkin-characteristic finite element method to numerically solve the convection-dominated problems in three space dimensions. The central idea in these methods is to rewrite the governing equations in terms of Lagrangian coordinates as defined by the characteristics associated with the