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$(1+s)$-Order Convergence Analysis of Weak Galerkin Finite Element Methods for Second Order Elliptic Equations

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Abstract. In this paper, we prove a lower convergence rate of the weak Galerkin finite element method for second order elliptic equations under a standard weak smoothness assumption. In all present literatures on weak Galerkin finite element methods for second order PDEs, the H^2 smoothness is compulsorily assumed for the real solution and hence a second order convergence is obtained. This lead to that the piecewise linear functions are excluded to construct finite element bases, although they behave very well in all numerical experiments. We intend to prove the $(1+s)$-order convergence rate under the H^1-smoothness assumption of the real solution and an additional $s>0$ regularity of the dual problem. Our strategy is that we firstly approximate the elliptic problem using the traditional finite element method with at least H^2 smooth bases, and then we apply the weak Galerkin method to approach this smooth approximating solution. Our result is an important supplementary for the weak Galerkin finite element method theory.

AMS subject classifications: 65N15, 65N30

Key words: Weak Galerkin methods, weak gradient operator, elliptic equation, convergence rate.

1 Introduction

The weak Galerkin (WG for short) finite element method is a recently developed innovative scheme for approximating partial different equations. The basic idea of WG method is in three aspects. Firstly, the partition of the domains is constructed by both polygons (or polyhedra for $d=3$) and all their edges. Secondly, the WG finite element basis is defined on either a polygon or an edge. At last, a weak version of differential operator can be defined in terms of WG finite element functions.

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The WG finite element method is first proposed in [17] by Wang and Ye for the second-order elliptic problem, in which the WG method is well studied and analyzed based on a discrete weak gradient operator arising from BDM elements [2] or RT elements [23]. The singularity of the weak gradient operator only admits the use of typical BDM or RT elements, which seriously limits the scope of the WG finite element method. Later on, to overcome this limitation, a stabilization term is introduced in [18, 19], and the WG finite element functions can be arbitrarily chosen as desired, which widely enlarges its application fields: for example, [45,46] for heat equation, [11,44] for a new WG method for general second-order elliptic problems, [36,37] for eigenvalue problems, [24] for elliptic interface problems with low regularity, [9, 12, 42, 43, 45] for the second order parabolic equation, [22] for the Helmholtz equation, [25, 27, 33, 39, 40] for Stokes equations, [32, 38] for Brinkman flow, [6, 13, 15] for Darcy-Stokes equation, [16] for Oseen equations, [10] for two-phase flow problem, [34] for incompressible fluid model, [21] for time-harmonic Maxwell's equations, [41] for biharmonic equation, [8] for Sobolev equation, [3, 4] for fourth order linear parabolic equation, [26] for a BDDC algorithm of the Stokes problem, [14] for singularly perturbed convection-diffusion–reaction problems, [31] for time dependent integro-differential equations, [30] for Cahn-Hilliard equation, [47, 48] for stochastic Parabolic PDE, etc. In these applications, the linear function bases behave very well in all computations.

The main idea of WG method is to define a weak differential operator in terms of the formula of integral by parts to replace the standard differential operators, such as the gradient, divergent, Laplace operators, etc. The finite element space for WG method is composed of discontinuous, piecewise polynomials which are respectively defined both in the interior and on the edges of any partition element. The weak differential operators are defined on a special discontinuous function space (weak function space), which is used to replace the standard finite element space. In [17,19,28], the authors derive a convergence order $\mathcal{O}(h^{k+1})$ for second-order elliptic equations on the premise that the exact solution is at least of H^{k+1} regularity for integer $k\geq 1$. To our knowledge, in all published literatures of WG finite element methods for second order problems, the H^{k+1} regularity are strictly demanded to assure the anticipative convergence rate. However, the domain Ω is usually assumed to be a polygonal or polyhedral for the WG finite element method, which implies the corresponding solution of a second order elliptic problem is just H^1-smoothness rather than H^2-smoothness. On the other hand, when we apply the traditional finite element method to approach a second order elliptic problem, we just require that the solution is of H^1 smoothness and the dual problem is of H^{1+s}-smoothness for some $0<s\leq 1$, then we can obtain the convergence rate $1+s$. Till now, under such weak smoothness assumption, the WG finite element method can not get any expected convergence rate for second order problems as a result of the use of trace inequality for dealing with stabilization term. This deduces that the basis of the WG finite element space requires that the polynomial order is of at least $(k+1)$, which theoretically excludes the choice of the linear function as WG basis. Actually, numerical experiments [8] show that the WG finite element method works very well when we choose linear functions to gen-