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A Novel B-Spline Method for Modeling Transport Problems in Anisotropic Inhomogeneous Media

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Abstract. In this paper, we present a new semi-analytical method based on the B-spline approximation for solving 2D time-dependent convection-diffusion-reaction equations to model transfer in the anisotropic inhomogeneous medium. The mathematical model is expressed as the initial-boundary value problem for quasi-linear parabolic equation with the second order elliptic spatial operator with mixed derivatives and variable coefficients. The time-stepping Crank-Nicolson scheme transforms the original equation into a sequence of quasi-linear elliptic partial differential equations. The approximate solution is sought as series over basis functions which are taken in the form of the tensor products of the B-splines with centers distributed inside the solution domain. Due to the modification of spline basis, the final approximate solution satisfies the boundary conditions of the initial problem with any choice of the coefficients of the series. The numerical examples demonstrate the high accuracy of the proposed method in solving 2D convection-diffusion-reaction problems in single- and multi-connected domains.

AMS subject classifications: 00A69

Key words: Convection-diffusion-reaction equation, anisotropic media, Crank-Nicolson scheme, B-spline, semi-analytical method.

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1 Introduction

The convection-diffusion-reaction equation (CDRE) is a mathematical model for the flow of heat, particles, or other physical quantities in various media. Let us consider the time-dependent CDRE as a balance condition [1,2]

$$\frac{\partial u}{\partial t}+\operatorname{div}(\mathbf{J}_a+\mathbf{J}_d)+s=0 \tag{1.1}$$

between the rate-of-change term, the influx through the boundary ∂D of the domain D, and the sources inside. The flux vector

$$\mathbf{J}_a=u\mathbf{v}(\mathbf{x},t)$$

describes the macro-transfer processes on the boundary, where $\mathbf{v}(\mathbf{x},t)=(v_1(\mathbf{x},t),v_2(\mathbf{x},t))$ is the velocity vector of the average movement of the medium and the variable of interest u denotes the transported quantity. The diffusive flux

$$\mathbf{J}_d=-\widehat{\mathbf{d}}(\mathbf{x},t)\nabla u$$

describes the micro/molecular transfer of the quantity u; the term s combines all other effects that create or destroy the quantity u. Throughout the paper we assume that the matrix of diffusion coefficients as follows

$$\widehat{\mathbf{d}}(\mathbf{x},t)=\begin{pmatrix} d_{1,1}(\mathbf{x},t) & d_{1,2}(\mathbf{x},t) \\ d_{2,1}(\mathbf{x},t) & d_{2,2}(\mathbf{x},t) \end{pmatrix},$$

which is a second order positive definite matrix for all $\mathbf{x}$ in the solution domain and $t\geq 0$. The source term $s=s(\mathbf{x},t,u,u_{x_1},u_{x_2})$ describes chemical reactions, heating, cooling, and similar processes and in general case is a nonlinear function of the arguments.

After some manipulations, Eq. (1.1) can be written

$$\frac{\partial u}{\partial t}=\mathcal{L}(\mathbf{x},t)[u]+f(\mathbf{x},t), \tag{1.2}$$

where

$$\begin{aligned}\mathcal{L}(\mathbf{x},t)[u]=&d_{1,1}(\mathbf{x},t)\frac{\partial^2 u}{\partial x_1^2}+2d_{1,2}(\mathbf{x},t)\frac{\partial^2 u}{\partial x_1\partial x_2}+d_{2,2}(\mathbf{x},t)\frac{\partial^2 u}{\partial x_2^2}\\&+b_1(\mathbf{x},t)\frac{\partial u}{\partial x_1}+b_2(\mathbf{x},t)\frac{\partial u}{\partial x_2}+q(\mathbf{x},t,u,u_{x_1},u_{x_2})\end{aligned} \tag{1.3}$$

is the second-order quasilinear operator with varying coefficients and mixed derivatives. The positive definite matrix $\widehat{\mathbf{d}}(\mathbf{x},t)$ provides the elliptic type of $\mathcal{L}$ and parabolic type of