Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **13**, No. 4, pp. 761-790

DOI: 10.4208/aamm.OA-2020-0044
August 2021

An Energy-Stable Finite Element Method for Incompressible Magnetohydrodynamic-Cahn-Hilliard Coupled Model

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Received 19 February 2020; Accepted (in revised version) 29 July 2020

Abstract. In this paper, we present an efficient energy stable finite element method for the two phase incompressible Magnetohydrodynamic (MHD) flow which is governed by the incompressible MHD equations and the Cahn-Hilliard equation. The strong nonlinear system governs the dynamics and the coupling of multiple physical fields which are, respectively, the velocity $\mathbf{u}$, the pressure p, the magnetic induction $\mathbf{B}$, the concentration ϕ, and the chemical potential μ. To solve the problem efficiently, we propose a linearized finite element scheme which is absolutely stable in time. Several numerical experiments are shown for demonstrating the competitive behavior of the method.

AMS subject classifications: 65N02, 65N12

Key words: Magnetohydrodynamic equations, Cahn-Hilliard equation, finite element method, absolutely energy-stable, constant auxiliary variable.

1 Introduction

The study of multi-phase flows has always been a concerned and difficult problem in fluid mechanics. It is also prevalent in many natural phenomena, such as bubble flow, that is bubbles of gas dispersed or suspended throughout the liquid continuous [1]. Of course in industry the large majority of processing technology involves multiphase flow. For example, gas-particles flow in combustion reactors and fiber suspension flows within the pulp and paper industry.

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Some multi-phase flows model are taken more attention to interface. Generally, interfaces separation can be considered a field that is divided into single phase regions by those interfaces or moving boundaries between phases [2–4]. The interface constitutive relations are presented in terms of the interface energy in both Lagrangian and Eulerian descriptions within the framework of finite deformation. Furthermore, if one of the flows is Magnetic nanofluid [5], the fluid flow and heat transfer may be controlled by an external magnetic field. We focus on the problem of multiphase flow with magnetic fluid [6]. Inspired by the Cahn-Hilliard (CH) and Navier-Stokes (NS) model [7–12], which described the interface variation in two-phase immiscible fluid, such as interface pinch-off, and moving contact lines, we proposed a phase-field model which consists of a Magnetohydrodynamic-Cahn-Hilliard (MHD-CH) system

$$\begin{cases} \phi_t+\mathbf{u}\cdot\nabla\phi-\mathtt{M}\Delta\mu=0, \\ -\Delta\phi+f(\phi)=\mu, \\ \mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-\gamma\Delta\mathbf{u}=\lambda\mu\nabla\phi+\mathtt{curl}\mathbf{B}\times\mathbf{B}+\mathbf{g}, \\ \mathtt{div}\mathbf{u}=0, \\ \mathbf{B}_t+\mathtt{curl}\Big(\mathbf{B}\times\mathbf{u}-\dfrac{1}{\sigma}\mathtt{curl}\mathbf{B}\Big)=0, \\ \mathtt{div}\mathbf{B}=0, \end{cases} \tag{1.1}$$

for $(\mathbf{x},t)\in\Omega\times(0,T)$ with $\Omega\subset R^3$ and a fixed time $T\in(0,+\infty)$. Here

$$f(\phi)=\frac{1}{\varepsilon^2}(\phi^2-1)\phi, \quad \text{which is } f(\phi)=\frac{\delta F(\phi)}{\delta\phi}, \quad F(\phi)=\frac{1}{4\epsilon^2}(\phi^2-1)^2.$$

Variables and parameters are listed in Table 1. We impose the initial and boundary conditions:

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}), \quad \mathbf{B}(\mathbf{x},0)=\mathbf{B}_0(\mathbf{x}), \quad \phi(\mathbf{x},0)=\phi_0(\mathbf{x}), \quad \forall\mathbf{x}\in\Omega, \tag{1.2a}$$

$$\begin{cases} \dfrac{\partial\phi}{\partial\mathbf{n}}=\dfrac{\partial\mu}{\partial\mathbf{n}}=\mathbf{0}, \\ \mathbf{u}=\mathbf{0} \quad \mathtt{on}\ \partial\Omega, \\ \mathtt{curl}\mathbf{B}\times\mathbf{n}=\mathbf{0}, \quad \mathbf{B}\cdot\mathbf{n}=0. \end{cases} \tag{1.2b}$$

The Cahn-Hilliard equation is a fourth order nonlinear parabolic partial differential equation, the variable μ is lead into the second equation in (1.1).

Considering the total energy, it contains kinetic energy $E_{\mathtt{kin}}$ (kinetic energy is the energy of an object because of its motion), magnetic energy $E_{\mathtt{mag}}$ (the energy that operates within a magnetic field) [13, 14], and surface free energy $E(\phi)$ (surface free energy is a sum of intermolecular interactions). And the total energy can be computed as follows

$$\begin{aligned} E_{\mathtt{tot}}&=\lambda E(\phi)+E_{\mathtt{kin}}+E_{\mathtt{mag}} \\ &=\lambda\Big[\frac{1}{2}\int_\Omega\|\nabla\phi\|_0^2d\mathbf{x}+\int_\Omega\frac{(\phi^2-1)^2}{4\varepsilon^2}d\mathbf{x}\Big]+\frac{1}{2}\int_\Omega\rho\|\mathbf{u}\|_0^2d\mathbf{x}+\frac{1}{2}\int_\Omega\frac{1}{\sigma}\|\mathbf{B}\|_0^2d\mathbf{x}. \end{aligned} \tag{1.3}$$