Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **13**, No. 4, pp. 791-805

DOI: 10.4208/aamm.OA-2020-0049
August 2021

Error Analysis of Two-Level Finite Element Method for the Nonlinear Conductivity Problem in Maxwell's System

Peizhen Wang[1,*], Dandan Zhang[2] and Wei Yang[3]

[1] *School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou, Henan 450046, China*
[2] *School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, Henan 450001, China*
[3] *Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Xiangtan, Hunan 411105, China*

Received 26 February 2020; Accepted (in revised version) 28 June 2020

Abstract. The traditional convergent analysis of two-level method (TLM) will fail when Nédélec finite element is employed to approximate Maxwell's system. In this paper, based on the superclose theory, we develop a new analysis framework for the nonlinear conductivity problem in Maxwell's system, which remedies the weakness of Nédélec finite element for two-level method. This method can save computational cost and improve the efficiency. We obtain the optimal convergent rate $\mathcal{O}(\Delta t+h^2)$ in spatial space. A numerical example verifies our theoretical analysis.

AMS subject classifications: 65N30, 65N15

Key words: Two-level method, nonlinear, conductivity, error estimates, superclose analysis.

---

1 Introduction

Two-level method is a numerical approximation based on the same mesh size to improve computational efficiency. Generally speaking, it is used to solve the problem with nonlinear terms. The basic idea is that we solve the given nonlinear problem on the constructed mesh by lower-order finite element, which is called the first level. This numerical solution will be taken as an approximation of the nonlinear term at next level. Then, at the second level, a linearized discrete scheme is established by the above result and handled by higher-order element on the same mesh, which will save computational cost, and thus improve the efficiency.

*Corresponding author.
Emails: peizhen86@126.com (P. Z. Wang), zdd@gs.zzu.edu.cn (D. D. Zhang), yangweixtu@126.com (W. Yang)

In this paper, we consider a general constitutive relation $\mathbf{J}=\mathbf{J}(\mathbf{E})$ for conductors in Maxwell's system

$$\begin{aligned}
&\epsilon\mathbf{E}_t-\nabla\times\mathbf{H}+\mathbf{J}(\mathbf{E})=\mathbf{f} && \text{in } \Omega\times(0,T], && (1.1a)\\
&\mu\mathbf{H}_t+\nabla\times\mathbf{E}=0 && \text{in } \Omega\times(0,T], && (1.1b)\\
&\mathbf{n}\times\mathbf{E}=0 && \text{on } \partial\Omega\times(0,T], && (1.1c)\\
&\mathbf{E}(\mathbf{x},0)=\mathbf{E}_0(\mathbf{x}), \quad \mathbf{H}(\mathbf{x},0)=\mathbf{H}_0(\mathbf{x}) && \text{in } \Omega, && (1.1d)
\end{aligned}$$

where Ω is a bounded convex polygon domain in $\mathbb{R}^3$ with a sufficiently piece smooth boundary $\partial\Omega$, and denotes the outward unit normal vector $\mathbf{n}$. $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{H}(\mathbf{x},t)$ stand for electric field and magnetic field, respectively. ϵ and μ represent the electric permittivity and magnetic permeability, respectively. The initial data $\mathbf{E}_0$ and $\mathbf{H}_0$ are the given function.

This system has appeared in many applications, such as microwave heating and specific applications where the electric conductivity switches on eddy currents in [1]. The power law of conductivity appeared in various physical models, for example, for type-II superconductors [2] and modelling of the nonlinear conductivity of the charge-density wave state of NbSe3 [3]. In [4], Maxwell's equations with nonlinear conductivity were studied with term $\mathbf{J}(\mathbf{E})=|\mathbf{E}|^{\alpha-1}\mathbf{E}$, where the constant $\alpha>0$. The authors started with a semi-discrete scheme in time and then proved the existence and uniqueness of the weak solution. Existence and uniqueness for the mixed nonlinear system were also proved in [5]. They proved the existence of the weak solution for a nonlinear function $\mathbf{J}(\mathbf{E})=\sigma(|\mathbf{E}|)\mathbf{E}$, with $\sigma(s)$ monotonically increasing. This article showed that the system had a unique weak solution. Moreover, the limit of the solution as $\epsilon\to 0$ converged to the solution of the quasi-stationary Maxwell's equations. In [6], the authors studied the time dependent eddy current equation, where the existence and uniqueness of weak solution was established in a proper Sobolev space. At the same time, a time discrete nonlinear scheme based on Rothe's method was designed and the convergence of approximating weak solution was proved. The similar Rothe's method was also applied to the nonlinear dielectric problem and magneto-heating problem [7,8].

Numerical analysis of nonlinear Maxwell's systems was also widely proposed in [4, 9–16]. In [4], the authors presented a numerical scheme to solve the coupled Maxwell's equations, where they considered the Back-Euler discreteness in the temporal and mixed conforming finite elements in the spatial space. In addition, a mixed finite element method for the Maxwell's equations with a nonlinear boundary was studied in [16]. In [15], a numerical scheme was derived to solve Maxwell's equations with a nonlinear conductivity in the form of a power law. In [9], the authors studied the numerical solution of time-dependent Maxwell's equations in three dimensional bounded polyhedral domain.

In order to reduce the calculation cost, TLM is one of the proper choices. Recently, in [17], the authors applied this method to simulate the Navier-Stokes problem. It consisted of solving one Navier-Stokes problem based on the P_1-P_1 finite element pair as the first level, and obtaining the optimal error estimates for the semi-discrete scheme. Then,