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Numerical Schemes for Time-Space Fractional Vibration Equations

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Abstract. In this paper, we present a numerical scheme and an alternating direction implicit (ADI) scheme for the one-dimensional and two-dimensional time-space fractional vibration equations (FVEs), respectively. Firstly, the considered time-space FVEs are equivalently transformed into their partial integro-differential forms with the classical first order integrals and the Riemann-Liouville derivative. This transformation can weaken the smoothness requirement in time when discretizing the partial integro-differential problems. Secondly, we use the Crank-Nicolson technique combined with the midpoint formula, the weighted and shifted Grünwald difference formula and the second order convolution quadrature formula to deal with the temporal discretizations. Meanwhile, the classical central difference formula and fractional central difference formula are applied to approximate the second order derivative and the Riesz derivative in spatial direction, respectively. Further, an ADI scheme is constructed for the two-dimensional case. Then, the convergence and unconditional stability of the proposed schemes are proved rigorously. Both of the schemes are convergent with the second order accuracy in time and space. Finally, two numerical examples are given to support the theoretical results.

AMS subject classifications: 65M06, 65M12, 35R1

Key words: Time-space fractional vibration equations, ADI scheme, stability, convergence.

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1 Introduction

In recent decades, it has been gradually found that fractional partial differential equations (FPDEs) play an increasingly important role in many fields of science and engineering, such as physics [16,30,42], biology [28,41], finance [29,33], hydrodynamics [1,20], and so on. However, as is known that it is always difficult or impossible to obtain the exact solutions of time-space FPDEs by using the existing analytical methods, see [27,31,32,43] for examples. Thus, there has been a growing interest to develop numerical methods for solving time-space FPDEs, see [4,6,9,10,15,18,19,24,39] and the references therein.

In this paper, we will consider the following time-space fractional vibration equations (FVEs) with the homogeneous initial boundary conditions

$$\frac{\partial^2 u(X,t)}{\partial t^2}+{}_0^{RL}\mathrm{D}_t^{\beta}u(X,t)=\Delta u(X,t)-(-\Delta)^{\frac{\alpha}{2}}u(X,t)+f(X,t),\quad 0<t\leq T, \tag{1.1}$$

where $1<\alpha,\beta<2$, $X\in\mathbb{R}^n$ with $n=1$ and 2, ${}_0^{RL}\mathrm{D}_t^{\beta}u(X,t)$ denotes the temporal Riemann-Liouville derivative with order β, i.e., for $n-1<\gamma<n$ with a positive integer n,

$${}_0^{RL}\mathrm{D}_t^{\gamma}u(X,t)=\frac{1}{\Gamma(n-\gamma)}\frac{\partial^n}{\partial t^n}\int_0^t(t-s)^{n-\gamma-1}u(X,s)ds.$$

Δ is the Laplacian operator. And $-(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian operator with order α, and it will be specifically defined in Subsections 3.1 and 4.1 for different dimensions. For the properties of Riemann-Liouville derivative and fractional Laplacian operator, one can refer [21,22] for examples. The model (1.1) is extended from [2,3], which can be used to describe the vibration of a string taking into account the friction in a medium with fractal geometry.

Intrinsically, Eq. (1.1) is a special form of multi-term time-space fractional wave equations (FWEs). Up to now, there exist many works on numerical methods for multi-term time-space FWEs, see [5,8,13,14,23,34,40] and the references therein. However, to the best of our knowledge, there is no existing numerical method which can be used to solve Eq. (1.1) directly. Thus, the purpose of this paper is devoted to constructing the high order schemes for the time-space FVEs (1.1) with one and two dimensions, and carrying out the corresponding numerical analysis for the proposed schemes.

It is well known that numerical methods for the integral equation are often more stable than the corresponding schemes derived from the equivalent differential equation. And discretizing an integral equation always requires the weaker smoothness than discretizing the equivalent differential equation to obtain the same numerical convergence order, see [12] for example. Thus, we in this paper firstly transform Eq. (1.1) into its equivalent partial integro-differential equation by using the integral operator. Secondly, the Crank-Nicolson technique is applied to deal with the temporal direction. Then, we use the midpoint formula to discretize the first order derivative, use the weighted and shifted Grünwald difference formula to discretize the Riemann-Liouville derivative, and