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High-Order Well-Balanced Finite Volume WENO Schemes with Conservative Variables Decomposition for Shallow Water Equations

Jiaojiao Li, Gang Li*, Shouguo Qian* and Jinmei Gao

School of Mathematics and Statistics, Qingdao University, Qingdao, Shandong 266071, China

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Abstract. This article presents well-balanced finite volume weighted essentially non-oscillatory (WENO) schemes to solve the shallow water equations (SWEs). Well-balanced schemes are characterized by preservation of the steady state exactly at the discrete level. The well-balanced property is of paramount importance in practical applications where many studied phenomena are regarded as small perturbations to equilibrium states. To achieve the well-balanced property, numerical fluxes presented here are constructed by means of a suitable conservative variables decomposition and the hydrostatic reconstruction idea. This decomposition strategy allows us to realize a novel simple source term approximation. Both rigorous theoretical analysis and extensive numerical examples all verify that the resulting schemes maintain the well-balanced property exactly. Furthermore, numerical results strongly imply that the proposed schemes can accurately capture small perturbations to the steady state and keep the genuine high-order accuracy for smooth solutions.

AMS subject classifications: 74S05

Key words: Shallow water equations, source term, WENO schemes, well-balanced property, hydrostatic reconstruction, conservative variables decomposition.

1 Introduction

The SWEs play a main role in the fields of hydraulic science and coastal engineering [1–3]. This article is concerned with developing high-order schemes for the SWEs over a non-

*Corresponding author.
Emails: jiaojiaojiao807@163.com (J. J. Li), gangli1978@163.com (G. Li), sgqian1976@163.com (S. G. Qian), gjinmei1979@163.com (J. M. Gao)

flat bottom topography, which in the one space dimension enjoy the following form

$$h_t+(hu)_x=0, \tag{1.1a}$$

$$(hu)_t+\left(hu^2+\frac{1}{2}gh^2\right)_x=-ghb_x, \tag{1.1b}$$

where $h(x,t)$, $u(x,t)$, $b(x)$ stand for the water depth, the depth-averaged flow velocity, and the bottom topography, respectively. The notations x and t are the space and time, while the real constant $g=9.812\mathrm{m/s}^2$ denotes the gravitational constant. The original system (1.1) can be rewritten into the following vector form

$$U_t+F(U)_x=S(U,b), \tag{1.2}$$

with $U=(h,hu)^T$, $F(U)=\left(hu,hu^2+\frac{1}{2}gh^2\right)^T$ and $S(U,b)=(0,-ghb_x)^T$ being the unknown conservative variables, the physical flux and the geometrical source term, respectively.

The system (1.1) belongs to the hyperbolic conservation laws (also referred as hyperbolic balance laws). Hyperbolic balance laws generally possess the form like (1.2), and have received growing consideration in the past decades. The balance laws usually allow the existence of non-trivial steady state solutions, in which the flux gradient is non-zero and is balanced by the geometricial source term exactly. Concerning the 1D SWEs, the well-known equilibrium state is the so-called still water steady state solutions

$$u=0, \quad h+b=\text{constant}. \tag{1.3}$$

The SWEs are highly nonlinear, such that analytical treatment on them is extremely difficult. Therefore, numerical computation has become a key means in studying SWEs. Furthermore, the above steady states and their small perturbations are of great interest in practical applications, and can not be commonly captured by standard schemes with direct treatment of the source term. In particular, this will lead to spurious oscillations even with very refined mesh. Moreover, the mesh refinement strategy will not eliminate the oscillations and only reduce the magnitude of the oscillations. Furthermore, such strategy is impractical due to high computational cost, especially for multi-dimensional problems.

To save the computational cost, well-balanced schemes [4, 5] have been proposed to exactly maintain the still water steady state up to machine accuracy at the discrete level. In comparison with the non-well-balanced counterpart, the well-balanced schemes can also resolve small perturbations of the steady state even on relatively coarse mesh [6, 7], and save the computational cost considerably, then improve the efficiency greatly. Following the pioneering works [4, 5], many researchers have attempted on this subject. Many representative research results include: kinetic scheme [8], gas-kinetic scheme [9], central-upwind scheme [10], finite volume evolution Galerkin method [11], WENO schemes [12–18], Hermite WENO scheme [19], central schemes [20, 21], discontinuous Galerkin (DG) methods [16, 22], ADER schemes [23, 24], spectral element method [25],