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Analytical Solution for Vibration of Continuously Varying-Thickness Beams Resting on Pasternak Elastic Foundations

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Abstract. In this paper, the vibration characteristics of beams with arbitrarily and continuously varying thickness and resting on Pasternak elastic foundations were analytically studied based on the elasticity theory directly. The general expression of stress function, which exactly satisfies the governing differential equations and the boundary conditions, was derived. The frequency equation governing the free vibration of varying-thickness beams resting on a Pasternak elastic foundation can be obtained by using the Fourier series expansion of the boundary conditions on the upper and lower surfaces of the beam. Convergence and comparison studies were conducted to demonstrate the high accuracy and efficiency of the present method. Application of the proposed analytical method to some typical beams with different geometry, Poisson's ratio, elastic coefficients of foundation were conducted further, and some new results are reported which may be used as an alternative of benchmark or standard solutions for numerical or other approximate results.

AMS subject classifications: 70J30, 74B05

Key words: Varying-thickness beam, free vibration, Pasternak elastic foundation, analytical solution, Fourier series expansion.

1 Introduction

Beam is one of the simplest structural units in engineering structures widely used in mechanical, civil and infrastructure fields and it is usually supported on special foundations. In this context, studies on mechanical behavior of beams on elastic foundations have attracted the concern of many researchers in various fields in the past decades.

Many kinds of foundation models have been proposed to describe the interaction between beams and foundations. The classical Winkler elastic foundation theory [1] has

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been widely used in engineering practice because of its simplicity. However, with Winkler model, the foundation is simplified as a series of independent springs, and the mutual relations between the springs are ignored, which is quite different from the engineering practice. For this reason, Pasternak [2] proposed a two-parameter model which takes the influence of shear stress in foundation in to account on the basis of Winkler elastic foundation theory. In the last two decades, many investigations have been condcuted on the bending [3–5], vibration [4–11], buckling [9–12] and modal analysis [13, 14], etc. of beams on Winkler or Pasternak elastic foundation. Recently, Mohanty et al. [15] investigated the free vibration characteristics of a two-layer elastic beam resting on a variable Pasternak foundation. Ghafarian et al. [16] used the differential transform method to study free and forced vibration of a Timoshenko beam on viscoelastic Pasternak foundation featuring bending-torsion.

Although there have been many studies on solutions for beams resting on elastic foundations up to date, most of their research objects are constant thickness beams. Compared with constant thickness beams, variable thickness beams have better structural characteristics and more optimized shapes as well as better economic performance, and have been more and more widely used in engineering. Available theoretical and numerical studies on variable thickness beams can be divided into two categories. One part of studies focused on beams with specific cross-section shapes, such as the single-tapered beams [8, 13, 17, 18], double-tapered beams [19], stepped beams [20], etc. The other part of studies investigated beams with non-uniform cross-section shape. For instance, Lee et al. [21, 22] derived the exact solutions for the problems governed by a general self-adjoint fourth-order nonhomogeneous ordinary differential equation with arbitrarily polynomial varying coefficients and general elastic boundary conditions in Green's function form. Recently, Zhao et al. [23] developed a new approach for free vibration of axially functionally graded Euler–Bernoulli and Timoshenko beams with non-uniform cross-section based on Chebyshev polynomials theory. Sınır et al. [24] used perturbation method and differential quadrature method to investigate the nonlinear free and forced vibration of axially functionally graded Euler-Bernoulli beams with non-uniform cross-section. Yuan et al. [25] proposed a novel method to simplify the governing equations for the free vibration of Timoshenko beams with both geometrical nonuniformity and material inhomogeneity along the beam axis and derived a series of exact analytical solutions from the reduced equations.

Based on the classical beam theories, different methods such as differential quadrature method [4, 5], differential transform method [16], Saint-Venant [26, 27], higher-order shear theories [28, 29], etc. were proposed, and in most cases the differential equation describing the problem has usually been simplified by introducing specific assumptions. Although these simplifications will reduce the solving complexity, the precision of solution will inevitably decrease correspondingly, which cannot be ignored in many modern engineering fields such as aerospace, apparatus and instrument microstructure, where high-precision solutions are often required. In this event, solutions directly from elasticity will have advantages than the classical simplified beam theories with higher precision.