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A Second-Order Energy Stable BDF Numerical Scheme for the Viscous Cahn-Hilliard Equation with Logarithmic Flory-Huggins Potential

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Abstract. In this paper, a viscous Cahn-Hilliard equation with logarithmic Flory-Huggins energy potential is solved numerically by using a convex splitting scheme. This numerical scheme is based on the Backward Differentiation Formula (BDF) method in time and mixed finite element method in space. A regularization procedure is applied to logarithmic potential, which makes the domain of the regularized function $F(u)$ to be extended from $(-1,1)$ to $(-\infty,\infty)$. The unconditional energy stability is obtained in the sense that a modified energy is non-increasing. By a carefully theoretical analysis and numerical calculations, we derive discrete error estimates. Subsequently, some numerical examples are carried out to demonstrate the validity of the proposed method.

AMS subject classifications: 65N30

Key words: Viscous Cahn-Hilliard, logarithmic potential, BDF scheme, error estimates.

1 Introduction

The Cahn-Hilliard (CH) equation which was first introduced by Cahn-Hilliard [1–3] describes the phase separation and coarsening phenomena in non-uniform systems such as alloys, glasses and polymer mixtures. The Cahn-Hilliard equation has been used as a model for various problems, whose applications are very extensive. We review some physical and industrial applications of Cahn-Hilliard model: microphase separation of diblock copolymers [4]; spinodal decomposition [5]; image inpainting [6]; phase-field modeling of tumor growth [7]; volume reconstruction [8]; topology optimization [9]; co-continuous binary polymer microstructures [10]; microstructures with elastic inhomogeneity [11], and multiphase fluid flows [12–14].

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Numerical methods for solving the Cahn-Hilliard equation provide an important tool for studying the dynamics described by the Cahn-Hilliard system, which have been extensively investigated. For the spatial discretization, a series of methods have been developed and applied. Finite difference methods and spectral methods [15–17] were proposed for rectangular regions. The finite element method can be used for the general domain of complex geometries [18–20]. Non-conforming elements or discontinuous Galerkin methods were proposed in [21–24]. For time discretization, energy stable methods have attracted more and more attention in the study of Cahn-Hilliard equations. Convex splitting method [25–27] is a very effective energy stable method, which is usually nonlinear. Stabilized semi-implicit methods [28–30] are linear schemes, which are also energy stable. A technique called Invariant Energy Quadratization (IEQ) which was successfully applied to different phase-field models by authors [31], was extended to handle the Cahn-Hilliard equation in [32, 33]. By introducing a scalar auxiliary variable (SAV), Shenjie [34] proposed a numerical technique to deal with nonlinear terms in gradient flows, which construct efficient and robust energy stable schemes for a large class of gradient flows [35, 36]. Exponential time differencing (ETD) method whose approximation is stabilized semi-implicit methods has been used in solving phase field equations [37, 38]. It is notice that the convex splitting method with nonlinear scheme is more accurate in comparison to above mentioned linear schemes.

The most of developed numerical algorithms mainly focused on the discretization of the polynomial potential for the Cahn-Hilliard equation. However, the free energy with the logarithmic potential is often considered to be more physically realistic than that with a polynomial free energy, because the former is derived from regular or ideal solution theories [39]. Dong and Wang et al. [40, 41] presented finite difference numerical scheme for the Cahn-Hilliard equation with a logarithmic Flory Huggins energy potential, which is unconditionally stabile and gave error estimates. Thomas P. Witelski [42] focused on the discussion of important qualitative features of the solutions of the nonlinear singular Cahn-Hilliard equation with degenerate mobility for the Flory-Huggins-deGennes free energy model. John and James [43] presented finite element method of the Cahn-Hilliard equation with a logarithmic free energy and non-degenerate concentration dependent mobility. Recently. Du et al. [44] discussed Allen-Cahn equation with logarithmic Flory-Huggins Potential based on ETD scheme, which could preserve maximum bound principle of Allen-Cahn type phase field equations and has applied to the Cahn-Hilliard [45] equation and epitaxial thin film equations [37]. More numerical methods for phase-field equations with logarithmic Flory-Huggins Potential can be also found in [46, 47].

Recently, researchers have devoted tremendous efforts to the relaxed Cahn-Hilliard equation, i.e., the viscous Cahn-Hilliard equation. Formally, the governing equation of the viscous Cahn-Hilliard equation is slightly different from the Cahn-Hilliard equation by incorporating one extra terms i.e., a strong damping (or viscosity) term. The viscous term was first proposed by Novick-Cohen [48] in order to introduce an additional regularity and some parabolic smoothing. It can be viewed as a singular limit of the phase field equations for phase transitions [49]. Significantly despite a great deal of