Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **13**, No. 4, pp. 892-913

DOI: 10.4208/aamm.OA-2020-0064
August 2021

Two-Grid Finite Volume Element Method Combined with Crank-Nicolson Scheme for Semilinear Parabolic Equations

Yuzhi Lou, Chuanjun Chen and Guanyu Xue*

School of Mathematics and Information Sciences, Yantai University, Yantai, Shandong 264005, China

Received 14 March 2020; Accepted (in revised version) 14 September 2020

Abstract. The aim of the paper is to propose a second-order accurate Crank-Nicolson scheme for solving semilinear parabolic equations. This scheme combing two-grid finite volume element method involves solving a small nonlinear system on a coarse grid space and a linear system on a fine grid space, which can improve computing efficiency while keeping the accuracy. It means that we can use large time steps in the actual calculation. We further prove the optimal error estimates of the scheme strictly and present numerous simulations to demonstrate the theoretical results.

AMS subject classifications: 65N12, 65M60

Key words: Two-grid method, finite volume element method, Crank-Nicolson scheme, error estimates, semilinear parabolic equations.

---

1 Introduction

We consider the following semilinear parabolic problem:

$$\begin{cases} u_t-\nabla\cdot(a(x)\nabla u)=f(x,t,u) & \text{in } \Omega\times(0,T], \\ u(x,t)=0 & \text{on } \partial\Omega\times(0,T], \\ u(x,0)=u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $x=(x_1,x_2)^T,\Omega\subset R^2$ is a bounded convex polygonal domain and $a(x)=(a_{ij}(x))_{i,j=1}^2$ is a given real-valued smooth function uniformly symmetric and positive definite in Ω. We assume that $f(x,t,u)$ is a real-valued function defined on $\Omega\times[0,T]\times R$ with the following growth-condition:

$$|f(x,t,w)-f(x,t,v)|\le C_f|w-v|(1+|w|+|v|)^{\gamma},\quad \forall w,v\in R, \tag{1.2}$$

---

*Corresponding author.
Emails: 1123097197@qq.com (Y. Lou), cjchen@ytu.edu.cn (C. Chen), gyxue@ytu.edu.cn (G. Xue)

where C_f is a positive constant and $0 \le \gamma < \infty$. Here for example $f(x,t,u)$ may be an arbitrary polynomial of u. The assumptions (1.2) is called mild-growth Lipschitz continuous or implies that f is locally Lipschitz continuous [1,2]. As in [2,3], we suppose that the initial data u_0 is sufficiently smooth and compatible and the problem (1.1) admits a unique solution satisfying

$$\max_{0\le t\le T} \|u(\cdot,t)\|_{3,q} + \|u_t(\cdot,t)\|_{3,r} \le M, \tag{1.3}$$

where M is a positive constant and $q,r \ge 1$ are constants to be specified in Section 3.

Finite volume element (FVE) method matains a lot of local conservation properties of physical quantities and has the advantages of simple structure and more flexible processing of complicated geometries. In the past three decades, the theoretical framework and basic tools of FVE method have developed rapidly [4–15]. For the linear case, Chou and Li [16] derived the error estimates in the L^2,H^1 and L^∞ norms. The finite volume method was applied for solving the parabolic problems and the corresponding error estimates were obtained in [17–19]. However, most studies focus on the right-hand term satisfying Lipschitz continuous condition. Afterwards, other numerical methods for parabolic problems were studied [20–23], such as local discontinuous Galerkin (LDG), discontinuous Galerkin immersed finite element (DG-IFE) etc. Gong et al. [24, 25] have studied finite element approximations for parabolic control problems.

Two-grid method was first proposed by Xu [26, 27] as a type of numerical tools for solving nonlinear equations. Meanwhile, Huang and Chen [28] proposed a multilevel iterative method that not only reduces the calculation but also preserves all of the high accuracy properties of finite element solutions singular problems, such as superconvergence, extrapolation etc. Afterwards, Dawson et al. [29, 30] applied two-grid method for nonlinear parabolic problem and proved the optimal H^1, L^2 estimates and superconvergence results. And then, this numerical method attracted great attention of scholars, for example, it was applied for linear and nonlinear elliptic problems in [31, 32]. A scheme combining the two-grid method with mixed finite element method was presented for the reaction-diffusion equations [33, 34]. In recent years, many scholars [35–42] further investigated two-grid method and obtained many good results. Crank-Nicolson scheme is one of the most classical methods for PDEs [43, 44]. For the problem (1.1), Chen et al. have proposed the first-order Backward Euler scheme by using the two-grid finite volume element method and made the corresponding convergence analysis in [45]. At the same time, Yang et al. [46] formulated and analyzed the post-processing FVE method which processed time at the last level. On this basis, we use the two-grid finite volume element method to construct the second-order Crank-Nicolson scheme at the time level, which will reduce the calculation cost while ensuring the accuracy.

In this paper, we propose finite volume element discretization based on Crank-Nicolson scheme for semilinear parabolic equations. The optimal error estimates of the method in H^1 and L^2 norms were proved. By using the two-grid method, we can obtain a rough approximation of the solution through solving a nonsymmetric and nonlinear