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A Two-Level Factored Crank-Nicolson Method for Two-Dimensional Nonstationary Advection-Diffusion Equation With Time Dependent Dispersion Coefficients and Source Terms

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Abstract. This paper deals with a two-level factored Crank-Nicolson method in an approximate solution of two-dimensional evolutionary advection-diffusion equation with time dependent dispersion coefficients and sink/source terms subjects to appropriate initial and boundary conditions. The procedure consists to reducing problems in many space variables into a sequence of one-dimensional subproblems and then find the solution of tridiagonal linear systems of equations. This considerably reduces the computational cost of the algorithm. Furthermore, the proposed approach is fast and efficient: unconditionally stable, temporal second order accurate and fourth order convergent in space and it improves a large class of numerical schemes widely studied in the literature for the considered problem. The stability of the new method is deeply analyzed using the $L^\infty(t_0,T_f;L^2)$-norm whereas the convergence rate of the scheme is numerically obtained in the L^2-norm. A broad range of numerical experiments are presented and critically discussed.

AMS subject classifications: 35K20, 65M06, 65M12

Key words: Two-dimensional advection-diffusion equation, time dependent dispersion coefficients, Crank-Nicolson approach, a two-level factored Crank-Nicolson method, stability and convergence rate.

1 Introduction and motivation

The advection-diffusion-reaction (ADR) equation still attracts research interests for its relevance to broad range of practical applications in environmental fluid mechanic, biology,

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chemistry and applied mathematics among other fields. This model is usually solved in the literature under the assumption that the dispersion coefficients are time and space independent. Some laboratory scale experiments suggest that in porous media transport problems, the dispersive parameter may be time-dependent [2, 3]. The factors that can affect this transport of pollutants include the solute properties, fluid velocity field within the subsurface and microgeometry such as the shape, size and location of the solid part of the medium or the layout of the voids. The transport equation often models flow in porous media, thermal pollution in river systems, dispersion of dissolved salts in groundwater, water transfer in soils, dispersion of tracers in subsurface, the spread of pollutants in rivers and streams, dispersion of dissolved material estuaries and costal seas, the absorbtion of chemical into the beds, contaminant dispersion in shallow lakes, forced cooling by fluids of solid material such as windings into turbo generators and long-range transport of solutes in the atmosphere [4, 9, 12, 34, 37, 38, 42]. This note deals with the two-dimensional unsteady transport equation with a first-order source/sink terms and time dependent dispersion coefficients describes by the following initial-boundary value problem

$$R\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left(\widetilde{D}_l\frac{\partial c}{\partial x}\right)+\frac{\partial}{\partial y}\left(\widetilde{D}_\tau\frac{\partial c}{\partial y}\right)-\widetilde{u}\frac{\partial c}{\partial x}-\widetilde{v}\frac{\partial c}{\partial y}-\mu Rc+\widetilde{q}, \quad (x,y)\in\Omega, \quad t\in(t_0,T_f], \tag{1.1}$$

with initial condition

$$c(x,y,t_0)=f(x,y), \qquad (x,y)\in\overline{\Omega}, \tag{1.2}$$

and boundary condition

$$c(x,y,t)=g(x,y,t), \qquad (x,y)\in\partial\Omega, \qquad t\in(t_0,T_f], \tag{1.3}$$

where $R>0$ is the retardation coefficient, $\widetilde{u}=\widetilde{u}(x,y,t)$ and $\widetilde{v}=\widetilde{v}(x,y,t)$ are longitudinal and lateral velocities, respectively, $\widetilde{D}_l=\widetilde{D}_l(t)$ and $\widetilde{D}_\tau=\widetilde{D}_\tau(t)$ denote longitudinal and transversal dispersion coefficients, respectively, $c(x,y,t)$ is the solute concentration, $\widetilde{q}=\widetilde{q}(x,y,t)$ and $\mu=\mu(x,y,t)$ represent the mass injection (or zero-order reaction rate coefficient) and first-order reaction rate, respectively. As defined in [2,39] we suppose that the functions $\widetilde{D}_l(t)$, $\widetilde{D}_\tau(t)$, $\widetilde{u}(x,y,t)$, $\widetilde{v}(x,y,t)$ and $\mu(x,y,t)$ are nonnegative and time variable increasing. t_0 is the initial time and T_f denotes the finite time. f and g are called initial and boundary conditions, respectively, $\Omega=(a_1,b_1)\times(a_2,b_2)$ and $\partial\Omega$ is the boundary of Ω. We assume that the initial and boundary conditions f and g, respectively, are regular enough so that the system of Eqs. (1.1)-(1.3), possesses a smooth solution.

In the last decades, many authors have applied various techniques in a search of analytic solutions of the nonstationary advection-diffusion equation with variable coefficients under suitable assumptions on the initial and boundary conditions satisfied by the unknown function [18, 39]. For problems which have not been covered by exact solutions, a broad range of numerical models have been deeply studied such as, explicit finite difference formulations, implicit finite difference techniques, implicit-explicit approaches, finite element and finite volume methods, cubic trigonometric B-splines procedures. We refer the readers to [5, 10, 13, 16, 21, 22, 24, 26, 33]. A reliable numerical