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An *h*-Adaptive Finite Element Solution of the Relaxation Non-Equilibrium Model for Gravity-Driven Fingers

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Abstract. The study on the fingering phenomenon has been playing an important role in understanding the mechanism of the fluid flow through the porous media. In this paper, a numerical method consisting of the Crank-Nicolson scheme for the temporal discretization and the finite element method for the spatial discretization is proposed for the relaxation non-equilibrium Richards equation in simulating the fingering phenomenon. Towards the efficiency and accuracy of the numerical simulations, a predictor-corrector process is used for resolving the nonlinearity of the equation, and an *h*-adaptive mesh method is introduced for accurately resolving the solution around the wetting front region, in which a heuristic *a posteriori* error indicator is designed for the purpose. In numerical simulations, a traveling wave solution of the governing equation is derived for checking the numerical convergence of the proposed method. The effectiveness of the *h*-adaptive method is also successfully demonstrated by numerical experiments. Finally the mechanism on generating fingers is discussed by numerically studying several examples.

AMS subject classifications: 65M60

Key words: Non-equilibrium Richard equation, *h*-adaptive mesh method, *a posteriori* error estimation, fingering phenomenon, porous media flow.

1 Introduction

To understand the mechanism of the preferential flow has been an important topic in the research area of the porous media flows. For example, in the case of infiltration of water in soil, the filtering capacity of the soil will be lost when infiltration occurs through a few

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preferential flow paths, which may cause serious disaster [29]. Several factors have been found for the formation of the preferential flow. See [7,21] for a comprehensive review.

There have been a number of works on understanding the mechanism of the formation of preferential flows. In one dimensional case, a phenomenon called saturation overshoot or pressure overshoot has been studied in detail [7,21]. From the plenty of the physical experiments, it has been found that the saturation overshoot caused the instability of the porous media flow [6]. Such conclusion has been successfully confirmed from the linear stability analysis based on non-equilibrium Richards equation [10,22]. In two dimensional case, the hold-back-pile-up effect [11] has been found to be an important mechanism for the formation of preferential flows. The hold-back operates at the forward edge of the wetting front to prevent overspreading due to the capillary diffusion, while the pile-up operates behind to increase the pressure and the water saturation of a finger tip [7].

Many models have been provided in the market for describing the porous media flows, from which the following two principal features can be concluded, i.e., i) the capability of generating initial unstable growth of small perturbations of the wetting front, and ii) the ability to limit lateral spreading behind the unstable front to keep the initially growing perturbations [10,21]. Due to its unconditional stability [10], the Richards equation (RE) can not be an appropriate model for describing the preferential flow. The relaxation non-equilibrium Richards equation (NERE) proposed in [15] is able to produce truly non-monotonic saturation profiles [9], by taking the dynamic (relaxation) effects with non-equilibrium pressure-saturation relation into account. Hassanizadeh et al. also showed in [20] that the hysteresis in the capillary pressure-saturation relation was a key point on the persistence of the fingers. In a number of physical experiments, the hysteresis phenomenon has been successfully observed, which describes a special relation between pressure and saturation. Consequently, in [2], Beliaev and Hassanizadeh developed a new model of capillary hysteresis by combining the hysteresis memory effects and dynamic (relaxation) effects together, and Schweizer [25] showed the mechanism and the importance of hysteresis in the study of preferential flows.

Towards the numerical methods for simulating porous media flows, a number of pioneering works have been available in the literature, for example, the finite difference methods [3,21,32,33], the discontinuous Galerkin methods [26], the finite element methods [16,17,24,30,34], etc. Besides, a convolutional neural network [12] is developed for rapidly predicting multiphase flow in heterogeneous porous media. For the profile of the finger, the water saturation around the finger tip is higher than that in the stationary finger core and a distribution layer [21]. Based on this observation, a uniformly fine mesh is not a smart choice for an efficient simulation. To balance the computational resources and the numerical accuracy, a discretization of the governing equation on a dynamically nonuniform mesh becomes a feasible way, which can be realized by using the adaptive mesh methods. The adaptive mesh techniques mainly include the r-adaptive methods [19] which redistribute the grid points while keeping the total number of mesh grids unchanged, the h-adaptive methods which locally refine and coarsen the mesh, and the