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Curve Optimization of Tapered Cantilever Beams Under Tip Loads

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Abstract. The study adopts the variational method for analyzing the cantilever tapered beams under a tip load as well as a definite end displacement, and further determining the optimized shapes and materials that can minimize the weights. Two types of beams are taken into account, i.e., the Euler-Bernoulli beam without considering shear deformation and the Timoshenko beam with shear deformation. By using the energy theorem and the reference of isoperimetric problem, the width variation curves and the corresponding minimum masses are derived for both beam types. The optimized curve of beam width for the Euler-Bernoulli beam is found to be a linear function, but nonlinear for the Timoshenko beam. It is also found that the optimized curve in the Timoshenko beam case starts from non-zero at the tip end, but its tendency gradually approaches the one of the Euler-Bernoulli beam. The results indicate that with the increase of the Poisson's ratio, the required minimum mass of the beam will increase no matter how the material changes, suggesting that the optimized mass for the case of Euler-Bernoulli beam is the lower boundary limit which the Timoshenko case cannot go beyond. Furthermore, the ratio ρ/E (density against Elastic Modulus) of the material should be as small as possible, while the ratio h^2/L^4 of the beam should be as large as possible in order to minimize the mass for the case of Euler-Bernoulli beam, of which the conclusion is extended to be applicable for the case of Timoshenko beam. In addition, the optimized curves for Euler-Bernoulli beam types are all found to be power functions of length for both tip point load cases and uniform load cases.

AMS subject classifications: 49K35, 49S05, 74P05

Key words: Tapered beam, Euler-Bernoulli beam, Timoshenko beam, variational principle, isoperimetric problem, curve optimization.

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1 Introduction

Tapered beams, also called non-prismatic beams, are beams with variable cross sections along the longitudinal directions in order to reduce their weights and adapt various stress distribution. Tapered beams are commonly used in many fields, such as airplanes, machinery, bridges and buildings. In a tapered beam, the bending stresses vary along the longitudinal axis and is no longer proportion to the bending moment since the section modulus also varies along the axis [11,14]. Applications of tapered beams under dynamic situations can be found in the literature [4,13]. A recent case is found in the literature for the tapered cantilever beam used in piezoelectric energy harvesters, which indicates the significance of tapered shape to provide the maximum efficiency [5].

Recent rapid developments of optimization technology and algorithms allows researchers to optimize tapered beams to a favorable level. Imam [7] explained the basic concept of shape optimization design and described the boundary effects. Dietl and Garcia [2] optimized the beam shapes for harvesting power using a heuristic optimization code, as well as experimental validation. Yoo [18] divided a tapered beam into multiple segments and assumed the width and the thickness as cubic spline functions to optimize. Ohsaki et al. [12], Katsikadelis et al. [8] and Kim et al. [9] optimized Euler-Bernoulli beams considering different load cases including torsional problems with the help of finite element method. Vinot et al. [16] presented a detailed procedure in shape optimization of thin-walled beam-like structures with correction coefficients. However, little literature can be found using variational method and considering shear effects in optimization, which is one of the objectives in this paper

The concept and application of variational methods are widely involved in many fields. The isoperimetric method, as a typical variational problem, has the elementary propositions to find the loop curve with a constant length covering the maximum area, or further determine the shape of a suspension cable. Such problems are normally turned into unconditional extremum problems using the Lagrange multiplier method and then solved using Euler-Lagrange Equation (transferring a variational problem to a calculus problem), see [1,3,6,10].

This study attempts to apply a variational method to optimize the tapering curve of a cantilever tapered beam under a tip load. Two different beam types are taken into considerations, i.e., the Euler-Bernoulli beam without shear deformation and the Timoshenko beam with shear deformation considered. The investigation will focus on the cantilever beam only tapered for the width of its cross section while keeping the same height of its cross section along the longitudinal axis, but the overall cross section is still kept as a rectangular shape (see Fig. 1). The results of this investigation could be utilized to optimize the tapering curve in order to minimize the mass with fixed tip displacements, e.g., for light-weight tapered beam designs in bridges and airplanes.