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A Fifth Order Alternative Mapped WENO Scheme for Nonlinear Hyperbolic Conservation Laws

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Abstract. In this work, we have developed a fifth-order alternative mapped weighted essentially nonoscillatory (AWENO-M) finite volume scheme using non-linear weights of mapped WENO reconstruction scheme of Henrick et al. (J. Comput. Phys., 207 (2005), pp. 542–567) for solving hyperbolic conservation laws. The reconstruction of numerical flux is done using primitive variables instead of conservative variables. The present scheme results in less spurious oscillations near discontinuities and shows higher-order accuracy at critical points compared to the alternative WENO scheme (AWENO) based on traditional non-linear weights of Jiang and Shu (J. Comput. Phys., 228 (1996), pp. 202–228). The third-order Runge-Kutta method has been used for solution advancement in time. The Harten-Lax-van Leer-Contact (HLLC) shock-capturing method is used to provide necessary upwinding into the solution. The performance of the present scheme is evaluated in terms of accuracy, computational cost, and resolution of discontinuities by using various one and two-dimensional test cases.

AMS subject classifications: 35L65, 65M08

Key words: High resolution scheme, unsteady, non-linear weights, numerical fluxes, alternative WENO scheme, hyperbolic equations.

1 Introduction

The development of high-order spatial accurate schemes for the solution of nonlinear hyperbolic equations is of great interest for the last few decades. The solution of hyperbolic equations contains discontinuities like rarefaction, shock wave, contact surfaces, and shear lines. These discontinuities are always present in the solution. Therefore, we need a high order scheme to capture these discontinuities without spurious oscillations. Weighted essentially non-oscillatory (WENO) and essentially non-oscillatory

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(ENO) schemes are widely used higher-order shock-capturing methods for solving nonlinear hyperbolic equations [1]. In the ENO scheme, the reconstruction of the convective flux is based on a fixed candidate stencil. In contrast, in the WENO scheme, reconstruction of the convective flux is done by the dynamic combination of different ENO stencils [2]. Harten et al. [3, 4] developed a high order ENO finite volume scheme, which is a modified version of the total-variation diminishing (TVD) scheme in [5]. Casper and Atkins [6], presented a high order non-oscillatory scheme for two-dimensional problems. Liu et al. [7] introduced a third-order WENO finite volume scheme. Jiang and Shu [8] constructed a 5th order WENO scheme denoted by WENO-JS. Later, Balsara and Shu [9] developed a very high-resolution WENO scheme (up to 11th order) in their work. WENO-JS can be made a finite volume or finite difference based scheme. The finite volume schemes are flexible and robust compared to the finite difference schemes [10]. Several finite volume WENO schemes have been designed and applied to curvilinear grid [11,12], non-uniform grids [13], unstructured grids [14] and structured grids [15–17]. The alternative WENO scheme developed by Shu and Osher [18] is an efficient scheme in the finite difference framework. It has been used with Cartesian [19] as well as curvilinear grids [20]. Liu [21] conducted a comparative study for the fifth-order alternative WENO scheme using different approximate Riemann solver for inviscid cases. The alternative methodology has also been used with the compact-WENO scheme [22], and the Hermite WENO scheme [23]. The alternative WENO approach could not perform with optimal accuracy at the points where derivatives become zero (critical points). Henrick et al. [24] designed the fifth-order mapped WENO scheme (WENO-M), in which mapping of nonlinear weights results in achieving 5th order accuracy at the critical points. Later WENO-Z scheme [25, 26] was developed, which provides better resolution and desirable convergence rate at the critical points. Recently, Wang et al. [27] studied an alternative finite difference WENO scheme with WENO-Z weights on a structured grid. Gao et al. [28] have extended the alternative WENO scheme framework to seventh and ninth order. This methodology has also been applied to shallow water equations, and multicomponent flows [29–31]. In the present work, we have designed a fifth-order alternative mapped WENO finite volume scheme using WENO-M nonlinear weights [24] for solving nonlinear hyperbolic equations. The proposed scheme is denoted by AWENO-M. It evaluates numerical flux using the reconstruction of primitive variables (solution variable) rather than conservative variables. The WENO reconstruction procedure based on conservative variables produces spurious oscillations as compared to primitive variables [32, 33]. The developed scheme results in small numerical oscillations near discontinuities (shock waves and contact surfaces) and gives the optimal rate of convergence at critical points. The shock-capturing HLLC method [34, 35] has been adopted for splitting the numerical flux due to its robustness. The performance of the resultant scheme is analyzed through different one and two-dimensional test problems. We assess the robustness of the proposed AWENO-M scheme by comparing computational cost and resolution of discontinuities with the traditional WENO scheme (WENO-JS) [8], WENO-M scheme [24], and the alternative WENO scheme based on WENO-JS nonlinear weights