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Numerical Methods for Semilinear Fractional Diffusion Equations with Time Delay

Shuiping Yang[1], Yubin Liu[1], Hongyu Liu[2,*] and Chao Wang[3,4,*]

[1] *School of Mathematics and Big Data Science, Huizhou University, Huizhou, Guangdong 516007, China*
[2] *Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong*
[3] *Department of Mathematics, Southern University of Science and Technology, Shenzhen, Guangdong 518055, China*
[4] *Guangdong Provincial Key Laboratory of Computational Science and Material Design, Southern University of Science and Technology, Shenzhen, Guangdong 518055, China*

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Abstract. In this paper, we consider the numerical solutions of the semilinear Riesz space-fractional diffusion equations (RSFDEs) with time delay, which constitute an important class of differential equations of practical significance. We develop a novel implicit alternating direction method that can effectively and efficiently tackle the RSFDEs in both two and three dimensions. The numerical method is proved to be uniquely solvable, stable and convergent with second order accuracy in both space and time. Numerical results are presented to verify the accuracy and efficiency of the proposed numerical scheme.

AMS subject classifications: 65N15, 65N30

Key words: Semilinear Riesz space fractional diffusion equations with time delay, implicit alternating direction method, stability and convergence.

1 Introduction

It is widely known that mathematical models with time delay are of fundamental importance in many scientific and engineering applications, including economics, physics, population ecology and medicine. As a result, the theoretical analysis and numerical computation of many differential equations with time delay have been studied by numerous researchers [1, 16, 23, 24]. Fractional differential equations with delay have received a lot of attentions [2,3,8,40,43,52,63,64] as a result of the development of fractional calculus in

*Corresponding author.
Emails: yang52053052@163.com (S. Yang), hongyliu@cityu.edu.hk (H. Liu), wangc3@sustech.edu.cn (C. Wang)

science and engineering [5,6,26,27,39,47–49]. As a typical example, the fractional Bloch equation with delay was proposed to depict the nuclear magnetic resonance [2]. Most of the problems for fractional differential equations and fractional differential equations with delay can not be solved analytically. Thereby, numerical treatment for such type of equations becomes a hot topic in the communities of numerical mathematics. In recent years, the attention on the numerical computations of fractional differential equations has been discussed by many researchers [7,10–14,19,20,22,42,51,54,55,57,60–62]. For example, in [59] the authors considered a class of variable order fractional advection diffusion equation with a nonlinear reaction term. The two-dimensional RSFDEs with nonlinear reaction term was studied in [21]. Recently, the alternating direction implicit Galerkin-Legendre spectral method was proposed to solve the two-dimensional nonlinear reaction-diffusion equations with the Riesz space-fractional derivatives in [53]. In [58], the authors developed the finite element method to solve the two-dimensional nonlinear Riesz space fractional derivatives Fisher′ s equation. In [56], a finite difference scheme was proposed for the two-dimensional diffusion equation with the Riesz space-fractional derivatives.

Very recently, some researchers developed methods on numerical solutions of fractional PDEs with time delay. The finite difference method was developed for solving the semi-linear space-fractional diffusion equations with time delay in [15]. In [45], the authors studied a linearized Crank-Nicolson method for solving the nonlinear fractional diffusing equation with multi-delay. In [41], the authors proposed the invariant subspace approach to solve a class of time-fractional partial differential equations with time delay. However, all of the works mentioned above focus on the one dimensional fractional PDEs with delay. In this paper, we shall develop high order schemes for the semilinear Riesz space-fractional diffusion equations with time delay in both two and three dimensions.

The rest of the paper is organized as follows. In Section 2, we present the numerical methods for the two-dimensional and three-dimensional semilinear RSFDEs with time delay. The stability and convergence of the method are proved in Section 3. Finally, we carry out some numerical experiments to confirm the theoretical results of the proposed method in Section 4.

2 Numerical methods for semilinear RSFDEs with time delay

In this paper, we consider the following two-dimensional and three-dimensional semilinear RSFDEs with time delay:

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} = K_x \dfrac{\partial^\alpha u(x,y,t)}{\partial |x|^\alpha} + K_y \dfrac{\partial^\beta u(x,y,t)}{\partial |y|^\beta} + f(x,y,t,u,u(x,y,t-s)), \\ \qquad\qquad\qquad\qquad 1<\alpha,\beta\le 2, \quad (x,y,t)\in\Omega\times[0,T], \\ u(x,y,t)=0, \qquad (x,y)\in\partial\Omega, \quad t\in[0,T], \\ u(x,y,t)=\varphi(x,y,t), \qquad (x,y,t)\in\Omega\times[-s,0], \end{cases} \tag{2.1}$$