Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **14**, No. 3, pp. 596-621

DOI: 10.4208/aamm.OA-2020-0400
June 2022

A Symmetric Inertial Alternating Direction Method of Multipliers for Elliptic Equation Constrained Optimization Problem

Mengyue Wu[1], Wenbao Ai[1], Jianhua Yuan[1,*] and Hui Tian[2]

[1] *School of Science, Beijing University of Post and Telecommunications, Beijing 100876, China*
[2] *State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing 100876, China*

Received 31 December 2020; Accepted (in revised version) 18 June 2021

Abstract. A new algorithm, called symmetric inertial alternating direction method of multipliers (SIADMM), is designed for separable convex optimization problems with linear constraints in this paper. The convergence rate of the SIADMM is proved to be $\mathcal{O}(1/\sqrt{k})$. Two kinds of elliptic equation constrained optimization problems, the unconstrained cases as well as the box-constrained cases of the distributed control and the Robin boundary control, are analyzed theoretically and solved numerically. First, the existence and uniqueness of the solutions to these problems are proved. Second, these continuous optimization problems are transformed into discrete optimization problems by the finite element method, and then the discrete optimization problems are solved by the proposed SIADMM. Numerical experiments with different problems are investigated to demonstrate the efficiency of the SIADMM. And the numerical performance of the SIADMM is better than the performance of the ADMM. Moreover, the numerical results show that the convergence rate of the SIADMM tends to be faster than $\mathcal{O}(1/\sqrt{k})$ in calculation process.

AMS subject classifications: 49M37, 65K10, 65K60, 90C25, 90C33

Key words: Symmetric inertial alternating direction method of multipliers, convergence rate, elliptic equation constraint, finite element method.

1 Introduction

An optimal control problem, which is to find the minimum value of an objective functional under some constraints, is called a PDE-constrained optimization problem while

*Corresponding author.
Email: jianhuayuan@bupt.edu.cn (J. Yuan)

the state constraint is specified by partial differential equations (PDEs). In the early 1970s, Jacques-Louis Lions made a pioneering work in the theory of the PDE optimal control problem [23]. Then, many scholars devote themselves to the research of theoretical analysis and numerical methods for this problem [1, 24, 26]. In recent years, the PDE optimal control problem has become an important research direction due to its wide application in engineering and other related fields, such as liquid flow, elastic deformation, option pricing, flow control, weather forecasting and so on [5, 7, 14, 17]. In practical problems, there is a challenge that how to improve the efficiency of methods for solving the large-scale PDE-constrained optimization problems.

Recently, the alternating direction method of multipliers (ADMM) was introduced into PDE-constrained optimization because of its simple implementation and high computational efficiency [32, 33]. And this method is widely used in image restoration and denoising, regularization estimation, statistics, deep learning and other fields [2,13,25,30, 31]. It takes the form of a decomposition-coordination procedure, in which the solutions of small local subproblems are coordinated to find a solution of a large global problem. This strategy is to splits the problem into several sub-problems, which reduces the computational cost compared to Newton-like methods. And this method can be viewed as an attempt to blend the benefits of dual decomposition and augmented Lagrangian methods for constrained optimization. Furthermore, many ADMM-like methods are proposed for the purpose of improving the ADMM [5, 6]. These ADMM-like methods usually have better numerical performances than the ADMM. More recently, a symmetric alternating direction method of multipliers (SADMM) was proposed for solving the separable convex optimization problems [20]. The numerical performance of the SADMM on solving optimization problems significantly outperforms the ADMM in both the CPU time and iteration number. In order to get a better performance in the numerical simulation, we improve the ADMM by drawing on the key technique of the SADMM.

In this paper, we design a new ADMM-like method called SIADMM to improve the numerical performance of the ADMM. The SIADMM is proving to be a convergent algorithm and the convergence rate of this method is $\mathcal{O}(1/\sqrt{k})$. In our work, optimal heat source problems are used to show the efficiency of the SIADMM, which are classical PDE-constrained optimization problems. Two kinds of elliptic equation optimal control problems, the unconstrained cases as well as the box-constrained cases of the distributed control and the Robin boundary control, are solved by a discretize-then-optimize approach. Examples of common discrete methods include: the finite difference method, the finite element method (FEM) and the mixed finite element method [8,11,12,18]. The existing results show that the FEM has higher calculation accuracy and advantages in solving problems posed on complex domains [27]. We use the FEM to transform continuous optimization problems into discrete optimization problems. However, the PDE constrained optimization problems will become large-scale problems when they are being discretized by the FEM. In order to overcome the difficulties brought about by solving large-scale problems, we use our proposed SIADMM to solve the discrete optimality systems.

For checking the efficiency of the SIADMM, we carry out several numerical experi-