Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **14**, No. 4, pp. 799-815

DOI: 10.4208/aamm.OA-2021-0364
August 2022

Ultra-Chaos: an Insurmountable Objective Obstacle of Reproducibility and Replication

Shijun Liao[1,2,3,*] and Shijie Qin[2]

[1] *State Key Laboratory of Ocean Engineering, Shanghai 200240, China*
[2] *Center of Marine Numerical Experiment, School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200240, China*
[3] *School of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China*

Received 6 December 2021; Accepted (in revised version) 19 December 2021

Abstract. In this paper, a new concept, i.e., ultra-chaos, is proposed for the first time. Unlike a normal-chaos, statistical properties such as the probability density functions (PDF) of an ultra-chaos are sensitive to tiny disturbances. We illustrate that ultra-chaos is widely existed and thus has general scientific meanings. It is found that statistical non-reproducibility is an inherent property of an ultra-chaos so that an ultra-chaos is at a higher-level of disorder than a normal-chaos. Thus, it is impossible in practice to replicate experimental/numerical results of an ultra-chaos even in statistical meanings, since random environmental noises always exist and are out of control. Thus, the ultra-chaos should be an insurmountable obstacle of reproducibility and replicability. Similar to Gödel's incompleteness theorem, such kind of "incompleteness of reproducibility" reveals a limitation of our traditional scientific paradigm based on reproducible experiments, which can be traced back to Galileo. The ultra-chaos opens a new door and possibility to study chaos theory, turbulence theory, computational fluid dynamics (CFD), the statistical significance, reproducibility crisis, and so on.

AMS subject classifications: 35N05, 65P20

Key words: Chaos, statistical stability, reproducibility and replication.

1 Introduction

The chaos theory [1–10] is widely regarded as the third greatest scientific revolution in physics in 20th century, comparable to Einstein's theory of relativity and the quantum mechanics. It is Poincaré [1] who first discovered "the sensitivity dependence on initial

*Corresponding author.
Email: sjliao@sjtu.edu.cn (S. Liao)

conditions" (SDIC) of chaotic systems, which was rediscovered by Lorenz [2] with a popular name "butterfly-effect": the exact time of formation and the exact path of a tornado might be influenced by tiny disturbances such as a distant butterfly that flapped its wings several weeks earlier. In addition, Lorenz [11] further discovered "the sensitivity dependence on numerical algorithms" (SDNA) of chaotic systems: computer-generated chaotic numerical simulations given by different algorithms in single/double precision quickly depart from each other with distinct deviations. Naturally, such kind of non-replicability of chaotic trajectory leaded to some heated debates on the credence of numerical simulations of chaos, and brought a crisis of confidence: some even made a rather pessimistic conclusion that "for chaotic systems, numerical convergence cannot be guaranteed *forever*" [12].

In order to gain a reproducible/reliable numerical simulations of chaotic trajectory, Liao [13] suggested a numerical strategy, namely the "Clean Numerical Simulation" (CNS). In the frame of the CNS [13–26], the temporal/spatial truncation errors are reduced to a required tiny level by means of a high *enough* order of Taylor expansion in time and a fine *enough* spatial discretization with spatial Fourier expansion, respectively. Besides, the round-off error is reduced to a required tiny level by means of reserving a large *enough* number of significant digits for all physical/numerical variables and parameters in multiple precision [27]. Furthermore, an additional simulation with even smaller numerical noises is needed to determine the so-called "critical predictable time" T_c by comparing these two simulations, so that the numerical noise is negligible and thus the computer-generated result is reproducible/reliable within the whole spatial domain in the time interval $t \in [0, T_c]$.

The CNS has been successfully applied to gain reproducible/reliable simulations of trajectories of many chaotic systems, such as Lorenz equation [16], Rayleigh-Bénard turbulent flows [18], and some spatiotemporal chaotic systems related to the complex Ginzburg-Landau equation [22], the damped driven sine-Gordon equation [23] and the chaotic motion of a free fall desk [24]. Especially, more than 2000 new periodic orbits of three-body system have been found by means of the CNS [19–21], which were reported twice by the popular magazine New Scientist [28, 29], because only three families of periodic orbits of the three-body problem have been reported in three hundred years since Newton mentioned the famous problem in 1687. All of these illustrate the validity of the CNS for chaos.

It should be emphasized that such a reproducible/reliable result given by the CNS provides us a "true" solution, and more importantly, a kind of *reproducibility* and *replicability* of chaotic trajectory. Such kind of trajectory reproducibility and replicability can remain in a long enough interval of time $t \in [0, T_c]$ as long as the background numerical noises could be globally reduced to a rather tiny level [22–24]. This kind of strict reproducibility and replicability of chaotic trajectory provides us a confidence of credence/reliability and especially a *benchmark* solution. For example, by means of an algorithm based on the CNS, Liao and Wang [16] gained a computer-generated chaotic simulation of Lorenz equation, which is reproducible/reliable in a quite long interval of