Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **14**, No. 4, pp. 893-913

DOI: 10.4208/aamm.OA-2020-0369
August 2022

Inverse Scattering Transform and Soliton Solutions for the Hirota Equation with N Distinct Arbitrary Order Poles

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Received 25 November 2020; Accepted (in revised version) 9 July 2021

Abstract. We employ the Riemann-Hilbert (RH) method to study the Hirota equation with arbitrary order zero poles under zero boundary conditions. Through the spectral analysis, the asymptoticity, symmetry, and analysis of the Jost functions are obtained, which play a key role in constructing the RH problem. Then we successfully established the exact solution of the equation without reflection potential by solving the RH problem. Choosing some appropriate parameters of the resulting solutions, we further derive the soliton solutions with different order poles, including four cases of a fourth-order pole, two second-order poles, a third-order pole and a first-order pole, and four first-order points. Finally, the dynamical behavior of these solutions are analyzed via graphic analysis.

AMS subject classifications: 35C08, 35Q15, 35Q51

Key words: The Hirota equation, zero boundary condition, Riemann-Hilbert problem, high-order poles, soliton solutions.

1 Introduction

In 1908, Plemelj made the affirmative answer to the Riemann-Hilbert (RH) problem with the help of Fredholm theory [1], turning the RH problem into the integral equation. Although this conclusion has always been flawed, the concepts and methods developed in the process of solving problems are still very useful. In 1972, Zakharov and Shabat extended the inverse scattering transform (IST) to the initial value problem of the classical nonlinear Schrödinger equation [2], and explained that this method is general, which laid a foundation for the study of other equations. In addition, the RH problem can be regarded as an extension of the IST, which can be used to study the integrable equations.

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It is not only used to analyze the long-time behavior of the initial value problem in the integrable system, but also has breakthrough applications in quantum mechanics and statistical mechanics models.

The Hirota equation can be written as

$$iu_t+\alpha u_{xx}+i\beta u_{xxx}+3i\gamma|u|^2u_x+\delta|u|^2u=0, \tag{1.1}$$

where u is a scalar function, α, β, γ and δ are real constants satisfying $\alpha\gamma=\beta\delta$ [3,4]. The exact N-soliton solution of the equation can be obtained by the Hirota bilinear method. By choosing $\delta=2\alpha$ and $\gamma=2\beta$, Eq. (1.1) can be reduced to

$$iu_t+\alpha(2|u|^2u+u_{xx})+i\beta(u_{xxx}+6|u|^2u_x)=0. \tag{1.2}$$

Eq. (1.2) is integrable shown in [5], because it can be regarded as the superposition of the nonlinear Schrödinger equation and the complex mKdV equation integrable flow. On the other hand, since it plays an important role in both physics and mathematics, hence the Hirota equation (1.2) has aroused widespread concern, and various works have been proposed. For example, the authors considered the initial boundary value of the Hirota equation (1.2) on the quarter plane in [6]. In [7], based on the RH problem, the authors studied the long-time asymptotic behavior of the Hirota equation (1.2) with initial value problems. In [8], the authors studied multi-solitons, breathers and rogue waves of Hirota equation (1.2) generated by Darboux transformation. In [9], the explicit soliton solution formula of the Hirota equation (1.2) is constructed by the IST proposed by Garderner, Greene, Kruskal and Miurra (GGKM). As far as we know, using the RH method to study the soliton solution of the Hirota equation with arbitrary order poles has not been reported. Compared with multiple high-order poles of NLS equation [42], Zhang mainly used two methods to prove that the denominator is regular. In this work, we mainly study the solution of a single high-order pole, the propagation behavior of the solution and the dynamic behaviors of the interaction of multiple high-order poles, and we find some rules, such as when the pole is taken as a special case, the soliton solution still retains the shape and direction before collision. Therefore in this work, we derive the exact solutions of the equation with arbitrary order poles via RH problem.

In the 1980s, the RH problem has been applied to the solution of integrable systems [10–12]. As the RH problem is proposed, the complex calculation process of the IST method is simplified to a great extent. It has gradually become the mainstream method for solving nonlinear integrable equations. The core idea is used to establish the corresponding RH problem, the relationship expression between the solution of the RH problem, and the solution of the initial value problem of the equation, via Lax pairs of integrable nonlinear evolution equation. In recent years, some excellent results have been obtained by using RH method, including Sasa-Satsuma equation [13], coupled derivative Schrödinger equation [14], coupled mKdV system [15], Wadati-Konno-Ichikawa equation [16], etc. [17–31]. For the two cases of simple zeros and high-order poles, we usually use the residue condition to directly solve the RH problem of simple poles and one high-order pole. But for the study of multiple high-order pole solitons in integrable systems,