Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **14**, No. 4, pp. 914-935

DOI: 10.4208/aamm.OA-2021-0018
August 2022

Lattice Boltzmann Model for Time-Fractional Nonlinear Wave Equations

Yibo Wang[1], Rui Du[1,*] and Zhenhua Chai[2]

[1] *School of Mathematics, Southeast University, Nanjing, Jiangsu 210089, China*
[2] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, China*

Received 25 January 2021; Accepted (in revised version) 30 May 2021

Abstract. In this paper, a lattice Boltzmann model with BGK operator (LBGK) for solving time-fractional nonlinear wave equations in Caputo sense is proposed. First, the Caputo fractional derivative is approximated using the fast evolution algorithm based on the sum-of-exponentials approximation. Then the target equation is transformed into an approximate form, and for which a LBGK model is developed. Through the Chapman-Enskog analysis, the macroscopic equation can be recovered from the present LBGK model. In addition, the proposed model can be extended to solve the time-fractional Klein-Gordon equation and the time-fractional Sine-Gordon equation. Finally, several numerical examples are performed to show the accuracy and efficiency of the present LBGK model. From the numerical results, the present model has a second-order accuracy in space.

AMS subject classifications: 65M10, 76M25

Key words: Lattice Boltzmann method, time-fractional wave equation, time-fractional Klein-Gordon equation, time-fractional Sine-Gordon equation.

1 Introduction

Fractional calculus is a suitable technique for describing materials with memory and genetic properties in the real world due to its nonlocality. In recent years, the fractional models have been widely used to describe anomalous diffusion [1–3], viscoelastic mechanics [4, 5], quantum mechanics and non-Newtonian fluid [6], fluid mechanics [7] and so on, since they are more general and usually more accurate than classical calculus. In fact, the time-fractional sub-diffusion equation and time-fractional super-diffusion equation can be used to model numerous natural phenomenon more accurately and more

*Corresponding author.
Emails: wangyb@seu.edu.cn (Y. Wang), rdu@seu.edu.cn (R. Du), hustczh@hust.edu.cn (Z. Chai)

authentically than their integer counterparts [8]. Mainardi et al. [9] point out that the propagation of the mechanic waves in viscoelastic media exhibits a power law creep, and this phenomenon can be depicted by the following time-fractional wave equation,

$$ {}_0^C\mathcal{D}_t^\kappa \phi(\boldsymbol{x},t) = D\nabla^2\phi(\boldsymbol{x},t) + F(\boldsymbol{x},t), \tag{1.1} $$

in which $\phi(\boldsymbol{x},t)$ is a field variable and $1<\kappa\leq 2$. ${}_0^C\mathcal{D}_t^\kappa\phi$ is the κ-order Caputo-type fractional derivative defined by

$$ {}_0^C\mathcal{D}_t^\kappa \phi(\boldsymbol{x},t) = \begin{cases} \dfrac{1}{\Gamma(2-\kappa)}\displaystyle\int_0^t (t-\varsigma)^{1-\kappa}\frac{\partial^2\phi}{\partial\varsigma^2}(\boldsymbol{x},\varsigma)d\varsigma, & 1<\kappa<2, \\ \dfrac{\partial^2\phi}{\partial t^2}(\boldsymbol{x},t), & \kappa=2, \end{cases} \tag{1.2} $$

where $\Gamma(\cdot)$ represents the Gamma function.

Compared with the classical differential equations, the fractional counterparts are usually more difficult to be solved analytically. In view of this, some numerical methods with high accuracy and efficiency have been developed in recent years, including finite difference method [10–13], finite element method [14–16], spectral method [17,18], series approximation method [19] and so on. Although these works have achieved fruitful results and greatly promoted the development of computational mathematics and engineering applications, they may not be easy to implement in high-dimensional case.

Following another path, the researches based on the kinetic theory has made great progress in the development of mesoscopic lattice Boltzmann method (LBM) for hydrodynamics [20–22], and this method has also been extended to solve the nonlinear differential equations [23–27]. Compared with the traditional numerical techniques, the LBM has some distinct features, such as natural parallelization, programming simplicity, computational efficiency and boundary implementation. Although the LBM has been used to study the nonlinear wave equations [28,29], only a little progress has been made to develop the efficient lattice Boltzmann models for the fractional systems. The following is a list of some existing LBM studies on fractional problems in recent years. Xia et al. [30] developed a novel multi-speed lattice Boltzmann numerical scheme for simulating the spatial fractional diffusion equation, which was an extension of the classical LBM. Wang et al. [31] presented a lattice Boltzmann approach for solving the convection-dispersion equation with the spatial fractional derivative, which was defined as a global integral from the initial to the current position. Zhou et al. [32] studied the anomalous diffusion phenomenon, wihch can be described by a superdiffusion equation with the Riemann-Liouville fractional derivative in space. In their work, a lattice Boltzmann model is constructed after transforming the fractional equation into a similar advection-diffusion equation. Cartalade et al. [33] established a multiple-relaxation-time lattice Boltzmann model for the advection-diffusion equation with a space-fractional derivative. Zhang et al. [34] developed a lattice Boltzmann model for the fractional diffusion equation with the Riemann-Liouville fractional derivative. Du et al. [35] transformed the sub-diffusion