Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **14**, No. 5, pp. 1059-1086

DOI: 10.4208/aamm.OA-2021-0095
October 2022

A New Fifth-Order Finite Volume Central WENO Scheme for Hyperbolic Conservation Laws on Staggered Meshes

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Received 31 March 2021; Accepted (in revised version) 16 August 2021

Abstract. In this paper, a new fifth-order finite volume central weighted essentially non-oscillatory (CWENO) scheme is proposed for solving hyperbolic conservation laws on staggered meshes. The high-order spatial reconstruction procedure using a convex combination of a fourth degree polynomial with two linear polynomials (in one dimension) or four linear polynomials (in two dimensions) in a traditional WENO fashion and a time discretization method using the natural continuous extension (NCE) of the Runge-Kutta method are applied to design this new fifth-order CWENO scheme. This new finite volume CWENO scheme uses the information defined on the same largest spatial stencil as that of the same order classical CWENO schemes [37,46] with the application of smaller number of unequal-sized spatial stencils. Since the new nonlinear weights are adopted, the new finite volume CWENO scheme could obtain the same order of accuracy and get smaller truncation errors in L^1 and L^∞ norms in smooth regions, and control the spurious oscillations near strong shocks or contact discontinuities. The new CWENO scheme has advantages over the classical CWENO schemes [37,46] on staggered meshes in its simplicity and easy extension to multi-dimensions. Some one-dimensional and two-dimensional benchmark numerical examples are provided to illustrate the good performance of this new fifth-order finite volume CWENO scheme.

AMS subject classifications: 65M60, 35L65

Key words: Finite volume scheme, central WENO scheme, NCE of Runge-Kutta method, staggered mesh.

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1 Introduction

In this paper, a new fifth-order finite volume central weighted essentially non-oscillatory (CWENO) scheme is designed for solving one-dimensional and two-dimensional hyperbolic conservation laws

$$\begin{cases} u_t+\nabla\cdot f(u)=0, \\ u(x,0)=u_0(x), \end{cases} \tag{1.1}$$

with suitable initial and boundary conditions on staggered meshes. Let us start by mentioning a few features and advantages of this new fifth-order finite volume CWENO scheme. The first is that one five-cell stencil and two two-cell stencils (in one dimension), or one twenty-five-cell stencil and four three-cell stencils (in two dimensions) are applied in the spatial reconstruction procedures. These new unequal-sized stencils are very different to the spatial stencils that adopted in the classical high-order finite volume CWENO schemes [37,46]. The second is that the number of these unequal-sized stencils is smaller than that of the same order classical CWENO schemes [36–39,46]. The third is that the largest spatial stencil of the new CWENO scheme is no bigger than that of the same order CWENO schemes [37,46], and this new CWENO scheme could get smaller truncation errors in L^1 and L^∞ norms in smooth regions on staggered meshes. The last is that the new nonlinear weights are proposed to maintain fifth-order accuracy in one and two dimensions on staggered meshes.

Hyperbolic conservation laws might contain the discontinuities via time approaching, even if the initial conditions are smooth enough. So the numerical solutions for hyperbolic conservation laws will also have the shock waves, contact discontinuities, and rarefaction waves in the computational field. The big challenge of the scholars is to design high-order numerical schemes to resolve different waves in smooth regions and to keep the sharp shock transitions in non-smooth regions. In 1984, Colella and Woodward [15] firstly designed a piecewise parabolic method (PPM) and employed a four-point centered stencil to define the interface value and then used them to limit spurious oscillations near strong discontinuities. Later, Leonard applied the limiting procedure with a high-order interface value in [35]. However, the proposed limiting procedures might degenerate the high-order accuracy to the first-order accuracy at smooth extrema. The PPM is an extension of the classical MUSCL scheme [60] and the MUSCL scheme is an extension of the Godunov's scheme [20]. To obtain the high-order of accuracy, Harten and Osher [24] proposed a weaker version of a TVD measurement [21] and gave a new basis for designing the high-order essentially non-oscillatory (ENO) schemes. In 1987, Harten et al. [23] used these ENO schemes to compute one-dimensional dynamic problems. Generally speaking, the most important idea of the ENO schemes is to use the nodal information of the smoothest stencil among all candidate stencils to approximate the variables at the half points or cell boundaries to obtain a high-order accuracy in smooth regions and keep essentially non-oscillatory property in non-smooth regions. In 1987, Harten [22] proposed a two-dimensional finite volume ENO scheme. In 1992,