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Positive and Conservative Characteristic Block-Centered Finite Difference Methods for Convection Dominated Diffusion Equations

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Abstract. In this work, spatial second order positivity preserving characteristic block-centered finite difference methods are proposed for solving convection dominated diffusion problems. By using a conservative piecewise parabolic interpolation with positive constraint, the temporal first order scheme is shown to conserve mass exactly and preserve the positivity property of solution. Taking advantage of characteristics, there is no strict restriction on time steps. The scheme is extended to temporal second order by using a particular extrapolation along the characteristics. To restore solution positivity, a mass conservative local limiter is introduced and verified to keep second order accuracy. Numerical examples are carried out to demonstrate the performance of proposed methods.

AMS subject classifications: 65M06, 76R99

Key words: Positivity preserving, conservative, characteristic method.

1 Introduction

Convection diffusion equations arise in mathematical modeling of many scientific and technical fields, such as atmospheric computation, oil reservoir simulation, groundwater and financial simulation [1,4,26,36]. Since there is usually no exact solution, accurate and reliable numerical methods for solving the problems are important for modeling phenomenons or studying the properties of systems that are governed by these equations.

For convection dominated diffusion equations, numerically solving the problems accurately is especially a challenging job. Due to the strongly hyperbolic nature of the problem, the solution often develops fronts that are nearly shocks, conventional finite

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difference methods or finite element methods usually cannot work well and show spurious non-physical oscillations (central difference/Galerkin scheme) or excessive numerical dissipations (upstream scheme) at steep fronts if the mesh size and time step size are not fine enough [24, 30]. The numerical examples exhibited in [13] and [17] show such drawbacks of several conventional numerical methods. Therefore, it is of great practical significance to develop effective methods for solving such a difficult problem.

In order to overcome the difficulty in solving convection dominated diffusion problem, a number of numerical schemes based on methods of characteristics are proposed. A modified method of characteristics is proposed by Douglas and Russell for solving one dimensional convection diffusion problems in [11]. Laterly, first order in time characteristics methods are further developed to solve high-dimensional problem [2, 5, 8, 10, 18, 23, 25, 30, 41, 43], and second order in time schemes are proposed in [14–16, 33]. From a physical point of view, the characteristic technique is natural, as the problem is solved effectively along the streamline and accurate approximation is obtained; from a computational point of view, methods of characteristics have much smaller time-truncation compared with the traditional finite difference methods. It is attractive because larger time steps size can be used, with corresponding efficiency improvement and without affecting accuracy, the computational cost can be reduced significantly [11]. Many researches have been devoted to the development of characteristics methods.

The block-centered method can keep second-order accuracy in space and it is a good choice for solving high gradient problems [34, 39]. A two-grid block-centered finite difference method for solving nonlinear parabolic integer-differential equations is given in [22]. For convection dominated diffusion equations, [41] uses the method of characteristics and proposed a block-centered finite difference method which is first-order accurate in time and second-order accurate in space.

Scientists and engineers often require approximations that accurately reflect physical reality. In many applications, unknowns represent quantities such as compound concentration [16, 19, 37, 40]. A typical requirement is to produce positive and conservative solution. It is known that the implement of method of characteristics needs approximation of the solution at tracking point in the previous time level. Standard high order interpolation schemes may break mass conservation and solution non-negativity. While using the technique of characteristics, the conservation property can be assured conveniently in a finite volume framework [20, 21]. However, designing a conservative characteristic method in the finite difference framework is nontrivial [7]. The weighted essentially nonoscillatory (WENO) schemes are a popular class of high-order accurate numerical methods for hyperbolic partial differential equations and other convection-dominated problems. A class of conservative semi-Lagrangian finite difference WENO schemes were developed by Qiu and Christlieb [31] and Qiu and Shu [32] for advection equations. Chen et al. [7] further developed a positive and conservative semi-Lagrangian finite difference WENO scheme for advection only problems. In order to achieve conservation, [6] developed a piecewise parabolic method (PPM) which uses a parabola interpolation that ensures local mass conservation for advection equations. Combining characteristic tech-