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High Order Efficient Algorithm for Computation of MHD Flow Ensembles

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Abstract. In this paper, we propose, analyze, and test a new fully discrete, efficient, decoupled, stable, and practically second-order time-stepping algorithm for computing MHD ensemble flow averages under uncertainties in the initial conditions and forcing. For each viscosity and magnetic diffusivity pair, the algorithm picks the largest possible parameter $\theta \in [0,1]$ to avoid the instability that arises due to the presence of some explicit viscous terms. At each time step, the algorithm shares the same system matrix with all J realizations but has different right-hand-side vectors for all J realizations. This saves assembling time and computer memory, allows the reuse of the same preconditioner, and can take the advantage of block linear solvers. For the proposed algorithm, we prove stability and convergence rigorously. To illustrate the predicted convergence rates of our analysis, numerical experiments with manufactured solutions are given on a unit square domain. Finally, we test the scheme on a benchmark channel flow over a step problem and it performs well.

AMS subject classifications: 65M12, 65M22, 65M60, 76W05

Key words: Magnetohydrodynamics, uncertainty quantification, fast ensemble calculation, finite element method, Elsässer variables, second order scheme.

1 Introduction

Numerical simulations of realistic flows are significantly affected by input data, e.g., initial conditions, boundary conditions, forcing functions, viscosities, etc, which involve uncertainties. As a result, uncertainty quantification (UQ) plays an important role in the validation of simulation methodologies and helps in developing rigorous methods to characterize the effect of the uncertainties on the final quantities of interest. A popular

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approach for dealing with uncertainties in the data is the computation of an ensemble average of several realizations. Many fluid dynamics applications e.g., ensemble Kalman filter approach, weather forecasting, and sensitivity analyses of solutions [14,41,45–47,56] require multiple numerical simulations of a flow subject to J different input conditions (realizations), which are then used to compute means and sensitivities.

Recently, the study of MHD flows has become important due to applications in e.g., engineering, physical science, geophysics and astrophysics [6, 9, 16, 19, 29, 57], liquid metal cooling of nuclear reactors [5,24,60], process metallurgy [15,58], and MHD propulsion [43,48]. For the time dependent, viscous and incompressible magnetohydrodynamic (MHD) flow simulations, this leads to solving the following J separate nonlinearly coupled systems of PDEs [8,15,39,50]:

$$u_{j,t}+u_j\cdot\nabla u_j-sB_j\cdot\nabla B_j-\nu\Delta u_j+\nabla p_j=f_j(x,t) \quad \text{in } \Omega\times(0,T], \tag{1.1a}$$

$$B_{j,t}+u_j\cdot\nabla B_j-B_j\cdot\nabla u_j-\nu_m\Delta B_j+\nabla\lambda_j=\nabla\times g_j(x,t) \quad \text{in } \Omega\times(0,T], \tag{1.1b}$$

$$\nabla\cdot u_j=0 \quad \text{in } \Omega\times(0,T], \tag{1.1c}$$

$$\nabla\cdot B_j=0 \quad \text{in } \Omega\times(0,T], \tag{1.1d}$$

$$u_j(x,0)=u_j^0(x) \quad \text{in } \Omega, \tag{1.1e}$$

$$B_j(x,0)=B_j^0(x) \quad \text{in } \Omega. \tag{1.1f}$$

Here, u_j, B_j, p_j, and λ_j denote the velocity, magnetic field, pressure, and artificial magnetic pressure solutions, respectively, of the j-th member of the ensemble with slightly different initial conditions u_j^0 and B_j^0, and forcing functions f_j and $\nabla\times g_j$ for all $j=1,2,\cdots,J$. The $\Omega\subset\mathbb{R}^d$ ($d=2$ or 3) is the convex domain, ν is the kinematic viscosity, ν_m is the magnetic diffusivity, s is the coupling number, and T is the simulation time. The artificial magnetic pressure λ_j are Lagrange multipliers introduced in the induction equations to enforce divergence free constraints on the discrete induction equations but in continuous case $\lambda_j=0$. All the variables above are dimensionless. The magnetic diffusivity ν_m is defined by $\nu_m:=Re_m^{-1}=1/(\mu_0\sigma)$, where μ_0 is the magnetic permeability of free space and σ is the electric conductivity of the fluid. For the sake of simplicity of our analysis, we consider homogeneous Dirichlet boundary conditions for both velocity and magnetic fields. For periodic boundary conditions or inhomogeneous Dirichlet boundary conditions, our analyses and results will still work after minor modifications.

To obtain an accurate, even classical Navier Stokes (NSE) simulation for a single member of the ensemble, the required number of degrees of freedom (dofs) is very high, which is known from Kolmogorov's 1941 results [40]. Thus, for a single member of MHD ensemble simulation, where velocity and magnetic fields are nonlinearly coupled, is computationally very expensive with respect to time and memory. As a result, the computational cost of the above coupled system (1.1a)-(1.1f) will be approximately equal to $J\times$ (cost of one MHD simulation) and will generally be computationally infeasible. Our objective in this paper is to construct and study an efficient and accurate algorithm for solving the above ensemble systems.