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Solving PDEs with a Hybrid Radial Basis Function: Power-Generalized Multiquadric Kernel

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Abstract. Radial Basis Function (RBF) kernels are key functional forms for advanced solutions of higher-order partial differential equations (PDEs). In the present study, a hybrid kernel was developed for meshless solutions of PDEs widely seen in several engineering problems. This kernel, Power-Generalized Multiquadric - Power-GMQ, was built up by vanishing the dependence of ϵ, which is significant since its selection induces severe problems regarding numerical instabilities and convergence issues. Another drawback of ϵ-dependency is that the optimal ϵ value does not exist in perpetuity. We present the Power-GMQ kernel which combines the advantages of Radial Power and Generalized Multiquadric RBFs in a generic formulation. Power-GMQ RBF was tested in higher-order PDEs with particular boundary conditions and different domains. RBF-Finite Difference (RBF-FD) discretization was also implemented to investigate the characteristics of the proposed RBF. Numerical results revealed that our proposed kernel makes similar or better estimations as against to the Gaussian and Multiquadric kernels with a mild increase in computational cost. Gauss-QR method may achieve better accuracy in some cases with considerably higher computational cost. By using Power-GMQ RBF, the dependency of solution on ϵ was also substantially relaxed and consistent error behavior were obtained regardless of the selected ϵ accompanied.

AMS subject classifications: 65D12, 65N35

Key words: Meshfree collocation methods, Radial Basis Function (RBF), partial differential equations (PDEs).

1 Introduction

Numerical methods such as Finite Element Method (FEM), Finite Volume Method (FVM), Boundary Element Method (BEM) and Meshless Methods has been extensively used to

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obtain the numerical solution of engineering problems [1]. As a result of many years of progress, FEM has become a standard method in the solution of differential equations [2]. Problems with complex domains have been successfully simulated with FEM, however, the process of mesh generation requires advanced algorithms and computational resources [3]. Since the mesh structure may have a considerable effect on the convergence and accuracy of FEM, the applications of meshless methods have been expanding owing to the fact that no mesh generation is needed. Radial Basis Functions (RBF) based methods is one type of the meshless methods and have been extensively investigated during the last two decades [4–8]. The radial characteristic of RBFs makes these functions easily applicable to higher dimensional problems and many of the RBFs are infinitely smooth. Kansa [9, 10] is the first to use the Multiquadric RBF in the solution of differential equations, since then, several type of RBFs such as Gaussian, Inverse Multiquadric, Bessel, Radial Powers and Wendland have been assessed in several engineering problems [11]. More details about various RBF approaches to solve differential equations can be found in the literature [12].

Exponential convergence rates were observed by using RBFs in the solution of differential equations [13]. However, the value of the shape/scale parameter, i.e., ϵ, is crucially important and an optimum value of the shape parameters usually exists. While more accurate results can be obtained for small values of the shape parameter, numerical instability and convergence problems in the solution step are generally observed [14]. Methods such as Contour-Pade [15], RBF-QR [16], HS-SVD [17] and WSVD for PU-RBF [18] were developed to overcome ill-conditioning. In another study, an improved Contour-Pade algorithm with vector-valued rational approximation method was implemented in RBF-FD to solve Poisson's equation in three-dimensional spherical shell [19]. These methods open up a low shape parameter-regime with more accurate results such that the precise determination of the shape parameter is not needed. Apart from these methods, preconditioning of the matrix [20], local support [21, 22] and hybrid kernels such as Gaussian-cubic kernel [23] were also used to decrease the condition number of the matrix in order to obtain a stable solution. All of these stable algorithms greatly advanced the implementation of RBF methods into the area of solving PDEs. However, these algorithms come with extra computational costs [23]. The knowledge about the efficiency of these algorithms on the solution of PDEs with complex domains is also substantially limited.

The optimum shape parameter, i.e., ϵ, value is highly influenced by several factors such as the differential equation, the RBF type, boundary conditions, node distribution and the number of nodes. The determination of the optimal ϵ is one of the subjects of the ongoing research [24–27]. Leave-one-out cross validation (LOOCV) [28] and maximum likelihood estimation [29] are two prominent methods used in RBF interpolation to estimate the optimum ϵ. However, these methods may not be effectively implemented for the determination of shape parameter in the solution of PDEs [30].

Moreover, polyharmonic spline RBFs such as Radial Powers and Thin Plate Splines are free of shape parameter which eliminates the disadvantage of finding an optimum