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A Well-Balanced Weighted Compact Nonlinear Scheme for Pre-Balanced Shallow Water Equations

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Abstract. It is well known that developing well-balanced schemes for the balance laws is useful for reducing numerical errors. In this paper, a well-balanced weighted compact nonlinear scheme (WCNS) is proposed for shallow water equations in pre-balanced forms. The scheme is proved to be well-balanced provided that the source term is treated appropriately as the advection term. Some numerical examples in one- and two-dimensions are also presented to demonstrate the well-balanced property, high order accuracy and good shock capturing capability of the proposed scheme.

AMS subject classifications: 65N12, 76M20

Key words: Shallow water equation, weighted compact nonlinear scheme, well-balanced property, shock capturing property.

1 Introduction

Shallow water equations (SWEs) with source terms are often used as the models to investigate practical issues that people are concerned about, such as tides, swells, dam breaks, and water pollution. The source terms may reflect the impact of external forces acting on fluid mass, like Coriolis acceleration, wind stresses, width change of channel or bottom, and lateral reactions [1]. In this paper, we confine the study to the case with source terms caused by bottom topographies. For instance, the one-dimentional case takes the following form

$$\begin{cases} h_t+(hu)_x=0, \\ (hu)_t+\left(hu^2+\dfrac{1}{2}gh^2\right)_x=-ghb_x, \end{cases} \tag{1.1}$$

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where h is the water height, u denotes the water velocity, g stands for the gravitational constant, and $b=b(x)$ represents the bottom topography.

It is well known that SWEs with source terms have an important balance law property, just like elastic wave equations, Euler equations with gravitational field, two phase flow models, etc. This property means that the source term can be exactly balanced by the flux gradient in a steady state. For example, a simple steady state for the one-dimensional SWEs (1.1) is defined by

$$h+b=H_0=\text{constant} \quad \text{and} \quad u=0. \tag{1.2}$$

Obviously, it is desirable to design numerical schemes to preserve this steady state exactly. The schemes with this property are called well-balanced schemes, which are useful for reducing numerical errors and hence for resolving small disturbances of steady state solutions. However, for many standard schemes used for hyperbolic conservation laws, this requirement is not satisfied.

The well-balanced property is also called the exact C-property, which was first proposed by Bermudez and Vazquez in [2], where an upwind scheme with this property was constructed for SWEs. Later on, Zhou [3] simplified this scheme by using a surface gradient method to deal with the source term. The new shallow water model was then named as a pre-balanced shallow water equation by Roges et al. in [4]. For this new model, a series of well-balanced schemes for SWEs have been developed recently [5–8]. For instance, the high order well-balanced weighted essentially non-oscillatory (WENO) schemes were proposed by Xing and Shu [9], using the method of spliting the source term. In [10], a well-balanced compact WENO scheme for SWEs based on the multi-resolution analysis of numerical solutions was constructed to detect discontinuities, such that the computational cost for nonlinear reconstruction is reduced. Instead of splitting the source term as done in [9], Li et al. [11] applied a linear finite difference operator to construct a WENO scheme for the pre-balanced shallow water model, resulting in an easily proved well-balanced scheme. More recently, Wen et al. developed in [12] a new entropy stable, well-balanced and positivity-preserving discontinuous Galerkin (DG) method for SWEs.

Most of the methods mentioned above are in the framework of WENO schemes or DG schemes. In this paper, we aim to develop a new well-balanced scheme for SWEs with source terms in the frame work of weighted compact nonlinear schemes (WCNSs) [13], which are often used in the field of computational fluid dynamcis. Compared with the original WENO scheme, WCNS adopts an interpolation procedure instead of the flux reconstruction procedure, resulting in superiority for solving problems on curvilinear grids, especially for freestream and vortex problems [14]. Recently, Gao et al. [15] developed a high order well-balanced WCNS for the SWEs. However, the interpolation is based on fluxes but not on variables. Thus a flux difference splitting is needed, as a result that the bottom function $b(x)$ has to be decomposed and reconstructed as the flux terms in a nonlinear way. More recently, Li et al. proposed in [16] a new well-balanced WCNS for SWEs, where interpolation for variables is adopted. However, one still have to split the source term technically to satisfy the well-balanced property. Different with the