Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **14**, No. 5, pp. 1201-1224

DOI: 10.4208/aamm.OA-2021-0231
October 2022

An Energy Stable SPH Method for Incompressible Fluid Flow

Xingyu Zhu[1], Shuyu Sun[1,*] and Jisheng Kou[2,3]

[1] *Computational Transport Phenomena Laboratory, Division of Physical Science and Engineering, King Abdullah University of Science and Technology, Thuwal 23955-6900, Saudi Arabia*
[2] *School of Civil Engineering, Shaoxing University, Shaoxing 312000, Zhejiang, China*
[3] *School of Mathematics and Statistics, Hubei Engineering University, Xiaogan 432000, Hubei, China*

Received 11 July 2021; Accepted (in revised version) 16 December 2021

Abstract. In this paper, a novel unconditionally energy stable Smoothed Particle Hydrodynamics (SPH) method is proposed and implemented for incompressible fluid flows. In this method, we apply operator splitting to break the momentum equation into equations involving the non-pressure term and pressure term separately. The idea behind the splitting is to simplify the calculation while still maintaining energy stability, and the resulted algorithm, a type of improved pressure correction scheme, is both efficient and energy stable. We show in detail that energy stability is preserved at each full-time step, ensuring unconditionally numerical stability. Numerical examples are presented and compared to the analytical solutions, suggesting that the proposed method has better accuracy and stability. Moreover, we observe that if we are interested in steady-state solutions only, our method has good performance as it can achieve the steady-state solutions rapidly and accurately.

AMS subject classifications: 65M12, 65M75, 76B03, 35Q30

Key words: Particle method, Smoothed Particle Hydrodynamics (SPH) method, incompressible fluid flow, energy stability, numerical stability.

1 Introduction

Smoothed Particle Hydrodynamics (SPH) is a particle-based mesh-free method to numerically solve fluid flow equations by discretizing the continuous fluid into a set of discrete particles. It was first presented to solve astronautic problems [1,2]. With an improvement in accuracy and efficiency, the SPH method has been applied to a wide range of research

*Corresponding author.
Emails: xingyu.zhu@kaust.edu.sa (X. Zhu), shuyu.sun@kaust.edu.sa (S. Sun), jishengkou@163.com (J. Kou)

and engineering fields, including computer graphics and computational fluid dynamics (CFD). In recent years, the SPH method has a number of attractive features as compared with traditional CFD methods [3, 4, 14]. First of all, it is a Lagrangian method, and the most significant benefit inherited from the Lagrangian method is the mesh-free property, which is suitable for solving the problems involving complicated geometry, high deformation and multiple components [5]. Other significant advantages include programmer-friendly implementation and low computational cost. In practice, the SPH method can be coupled with traditional mesh-based methods like the Finite Difference Method (FDM) and Discrete Element Method (DEM) to balance accuracy and efficiency [6].

A number of recent researches on the SPH methods focus on the incompressible fluid flow, and in many scenarios the incompressibility is an essential property of the fluid. The weakly compressible SPH (WCSPH) method is first used to solve the incompressible fluid flow, where the incompressibility is approximated by the weak compressibility [8,9]. In the WCSPH method, the velocity unknown is updated explicitly, and the pressure is related to the density and calculated by an artificial state equation. The incompressibility constraint is treated by high sonic velocities and small time steps. The WCSPH method cannot guarantee that the fluid is completely incompressible, but it has the merits of being easy to implement and computationally efficient when the number of particles is small.

In order to obtain more accurate results, the Incompressible SPH (ISPH) method is proposed for the incompressible fluid flow simulation [10]. According to the ISPH method, the pressure field can be determined by a projection scheme in which a Pressure Poisson equation (PPE) needs to be solved. The most popular ISPH methods include the Implicit Incompressible SPH (IISPH) [11], Predictive-Corrective Incompressible SPH (PCISPH) [12], Divergence-Free SPH (DFSPH) [13] and corrective methods based on these three methods. The projection scheme is usually used in the ISPH methods. It provides a semi-implicit procedure that enables us to obtain the pressure by solving the PPE [12]. Generally, the PPE can be derived from either a constant density constraint or a divergence-free velocity constraint. On the one hand, the projection scheme calculates the pressure implicitly, which typically leads to higher performance and allows large time steps. On the other hand, the constraint ensures the density deviation desired or velocity field divergence-free, resulting in a more accurate and stable simulation in the incompressible fluid flow. In general, the ISPH methods show the advantages of greater accuracy and stability. However, in the existing projection methods used for the ISPH methods, the velocity representation in the PPE is not mathematically consistent with the one in the equation for updating velocity. This violation of consistency in the methods might increase the numerical instability.

The laws of mass and momentum conservation have been widely discussed in the incompressible SPH analysis, while the energy conservation law was usually neglected because the energy change was considered to be very small in the incompressible fluid flow simulation. However, energy conservation has a significant impact on the numerical stability of the SPH methods [15]. The stability analysis generally consists of two aspects,