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Deep Domain Decomposition Methods: Helmholtz Equation

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Abstract. This paper proposes a deep-learning-based Robin-Robin domain decomposition method (DeepDDM) for Helmholtz equations. We first present the plane wave activation-based neural network (PWNN), which is more efficient for solving Helmholtz equations with constant coefficients and wavenumber k than finite difference methods (FDM). On this basis, we use PWNN to discretize the subproblems divided by domain decomposition methods (DDM), which is the main idea of DeepDDM. This paper will investigate the number of iterations of using DeepDDM for continuous and discontinuous Helmholtz equations. The results demonstrate that: DeepDDM exhibits behaviors consistent with conventional robust FDM-based domain decomposition method (FDM-DDM) under the same Robin parameters, i.e., the number of iterations by DeepDDM is almost the same as that of FDM-DDM. By choosing suitable Robin parameters on different subdomains, the convergence rate is almost constant with the rise of wavenumber in both continuous and discontinuous cases. The performance of DeepDDM on Helmholtz equations may provide new insights for improving the PDE solver by deep learning.

AMS subject classifications: 35Q68, 65N55, 78A48

Key words: Helmholtz equation, deep learning, domain decomposition method, plane wave method.

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1 Introduction

The Helmholtz equation, the main research object in this paper, is a kind of important partial differential equation (PDE) in scientific computing and industrial applications. It can be regarded as a time-independent form of the wave equation, which has important applications in acoustics and describes the propagation of waves. Besides acoustic waves, it is also used to describe electromagnetics because it can be reduced from Maxwell's equations [26]. Moreover, in the fields of elasticity, quantum mechanics and geophysics, we also encounter different forms of Helmholtz equations.

We assume that there is a bounded Lipschitz domain $\Omega\subset\mathbb{R}^d$, $d=2,3$. In addition to the true physical part, the domain Ω may also contain artificial layers, for example, representing perfectly matched layers [4]. A general Helmholtz equation is given in the following

$$\mathcal{L}u := -\nabla^T(\alpha\nabla u) - \frac{\omega^2}{\kappa}u = f \qquad \text{in } \Omega, \tag{1.1a}$$

$$\mathcal{B}u = g \qquad \text{on } \partial\Omega, \tag{1.1b}$$

where $\omega\in\mathbb{C}$ and the coefficient matrix α, the scale field κ and the source term f are all given complex-valued functions. The boundary condition $\mathcal{B}u=g$ can be one or a combination of the following cases,

$$u = g_D \qquad \text{on } \Gamma_D, \tag{1.2a}$$

$$\mathbf{n}^T(\alpha\nabla u)+p_0u = g_R \qquad \text{on } \Gamma_R, \tag{1.2b}$$

$$\mathbf{n}^T(\alpha\nabla u)+p_0u+p_1\mathbf{n}^T(\alpha\nabla_S u)-\nabla_S^T(q_1\Pi_S(\alpha\mathbf{n})u+p_2\alpha\nabla_S u) = g_V \qquad \text{on } \Gamma_V, \tag{1.2c}$$

where $\mathbf{n}$ is the unit outer normal vector, ∇_S is the surface gradient, p_0, p_1, p_2, q_1 are complex-valued functions and Π_S is the orthogonal projection onto the tangential plane of the surface. In order to simplify the notations and facilitate discussion, we will use a simpler Helmholtz equation version that usually is seen in other papers.

We look for the numerical solution of the heterogeneous Helmholtz equation as follows,

$$-\Delta u - k^2(x)u = f \qquad \text{in } \Omega, \tag{1.3a}$$

$$\frac{\partial u}{\partial \mathbf{n}} + \mathbf{i}k(x)u = g_a \qquad \text{on } \partial\Omega, \tag{1.3b}$$

$$[u] = u^+ - u^- = g_b \qquad \text{on } \Gamma, \tag{1.3c}$$

$$\left[\frac{\partial u}{\partial \mathbf{n}}\right] = \left(\frac{\partial u}{\partial \mathbf{n}}\right)^+ - \left(\frac{\partial u}{\partial \mathbf{n}}\right)^- = g_c \qquad \text{on } \Gamma, \tag{1.3d}$$

where $\mathbf{n}$ is the unit outer normal vector on the boundary $\partial\Omega$ or the interface Γ, $k(x)>0$ is the wavenumber. In practice, $k(x)$ would be the continuous variable coefficient or piecewise constants, not just a constant. The two formulas (1.3c) and (1.3d) describe jump conditions of u and $\frac{\partial u}{\partial \mathbf{n}}$ on the interface Γ. Actually, the numerical solution of the Helmholtz