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The Discontinuous Galerkin Method by Patch Reconstruction for Helmholtz Problems

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Abstract. This paper develops and analyzes interior penalty discontinuous Galerkin (IPDG) method by patch reconstruction technique for Helmholtz problems. The technique achieves high order approximation by locally solving a discrete least-squares over a neighboring element patch. We prove a prior error estimates in the L^2 norm and energy norm. For each fixed wave number k, the accuracy and efficiency of the method up to order five with high-order polynomials. Numerical examples are carried out to validate the theoretical results.

AMS subject classifications: 35J05, 65N30

Key words: Least-squares reconstruction, Helmholtz problems, patch reconstruction, discontinuous Galerkin, error estimates.

1 Introduction

The Helmholtz equation is a linear mathematical model that describes time-harmonic acoustic, elastic and electromagnetic steady state waves. One major problem in approximating this equation by classical finite element methods is the loss of ellipticity with an increasing excitation frequency.

For many years, the finite element method (and other type methods) has been widely used to discretize the Helmholtz equation with various types of boundary conditions, see [1,5–7,10,18,19,22,29] and the references therein. It is well known that, in every coordinate direction, one must put some minimal number of grid points in each wave length

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$l = 2\pi/k$ in order to resolve the wave; that is, the mesh size h must satisfy the constraint $hk \lesssim 1$. In practice, 6-10 grid points are used in a wave length, which is often referred to as the "rule of thumb". However, this "rule of thumb" was probed rigorously not long ago by Ihlenburg [19] only in the one-dimensional case (called the preasymptotic error analysis). The main difficulty of the analysis is caused by the strong indefiniteness of the Helmholtz equation, which in turn makes it hard to establish stability estimates for the finite element solution under the "rule of thumb" mesh constraint. Standard finite element methods based on low-order polynomials do not perform well for the Helmholtz equation at high wavenumber. On the one hand, low-order polynomials do not well resolve the solution unless several grid points per wave length are used. On the other hand, such methods suffer from the so-called pollution effect: for a fixed number of grid points per wave length, the numerical error grows with the wavenumber [20, 21]. The detailed analysis of [15, 19] also shows that the pollution effect is inherent in the finite element method and is caused by the deterioration of stability of the Helmholtz operator as the wave number k becomes large. Indeed, it was suggested in [27, 28] that the pollution effect can be suppressed by using higher order polynomials for problems with higher wavenumber.

In the past fifteen years, DG methods have received a lot of attention and undergone intensive studies by many people. We refer the reader to [3, 4, 8, 12–14, 30, 32] and the references therein for a detailed account on DG methods for coercive elliptic and parabolic problems. We like to note that, in addition to the well known advantages of DG methods, the results of this paper also demonstrate the flexibility and effectiveness of DG methods for strongly indefinite problems, which was not well understood before. In this article, the discontinuous Galerkin method by patch reconstruction will be employed to study the Helmholtz problem. The method is an efficient numerical method for solving partial differential equations, was firstly introduced in [24] for the elliptic problems, and applied to many other model problems [23, 25, 26]. In [24], Li et al. proposed an arbitrary-order discontinuous Galerkin method for second-order elliptic problem on general polygonal mesh with only one degree of freedom per element by solving a local discrete least-squares over a neighboring element patch. In this work, our mesh-dependent sesquilinear forms penalize the jumps of the function values across the element edges/faces.

This paper has a small but vitally important idea that takes the penalty parameters as complex numbers of positive imaginary parts. This idea also contributes critically to the stability of the IPDG methods of this paper. The rest of the paper is organized as follows. In Section 2, we briefly describe the reconstruction finite element space, then state the basic properties of those spaces. In Section 3, we present the interior penalty discontinued Galerkin method for Helmholtz problem with the reconstructured approximation space and prove a priori error estimate. In Section 4, we perform several benchmark problems for Helmholtz problem to demonstrate the efficiency of the proposed method. Finally, in Section 5, we summarize the work and draw some conclusions.