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Radially Symmetrical Problems for Compressible Fluids with a High-Resolution Boundary Condition

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Abstract. Imposing appropriate numerical boundary conditions at the symmetrical center $r=0$ is vital when computing compressible fluids with radial symmetry. Extrapolation and other traditional techniques are often employed, but spurious numerical oscillations or wall-heating phenomena can occur. In this paper, we emphasize that because of the conservation property, the updating formula of the boundary cell average can coincide with the one for interior cell averages. To achieve second-order accuracy both in time and space, we associate obtaining the inner boundary value at $r=0$ with the resolution of the corresponding one-sided generalized Riemann problem (GRP). Acoustic approximation is applied in this process. It creates conditions to avoid the singularity of type $1/r$ and aids in obtaining the value of the singular quantity using L'Hospital's rule. Several challenging scenarios are tested to demonstrate the effectiveness and robustness of our approach.

AMS subject classifications: 65M08, 76L05, 76M12, 76N15

Key words: Radially symmetrical, high-resolution, conservation; singularity, numerical boundary condition, GRP, acoustic approximation.

1 Introduction

The study on compressible fluid flows with the feature of radial symmetry touches upon many disciplines in engineering and science [4,7,13]. Numerically speaking, many common flow problems, such as the implosion or explosion in air, the coronal transient and the bullet-shooting can be simply formulated and effectively solved in the radially symmetrical coordinates [23,28]. Specifically, the radially symmetrical problems can be divided into two categories: cylindrically and spherically symmetrical ones.

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For either category, we must overcome the difficulty of geometrical singularity, impose appropriate numerical boundary conditions, and address the resolution of classical discontinuities. Only a few related discussions on radially symmetrical fluids have been documented over the past years. In [11], the generalized Riemann problem (GRP) method was used to track discontinuities. However, it primarily focused on the geometrical effect on numerical fluxes. In [25], a modified Harten's TVD scheme was devised by reducing the two-dimensional scheme in Cartesian coordinates to the one in polar coordinates. A series of cell-centered schemes were proposed in [6,17,24,26]. By compatibly discretizing the source term in the momentum equation, they were capable to preserve the obligatory symmetry and conservation properties.

Most previous schemes did not fully conduct systematic analyses of numerical boundary conditions at the symmetrical center $r=0$ and its embedding into numerical schemes. Nevertheless, an improper approach may result in numerical oscillations and even nonphysical solutions [29]. In [12], the Cartesian coordinates were specially inserted near $r=0$, while the cylindrical coordinates were used in the outer region. Excess complications of connecting two regions can arise. In [22], the Jacobian factor of coordinate transformation was applied to remove the singularity of $1/r$. The subsequent analysis was based on the assumption that the boundary cell is exceedingly smaller than the interior ones. Other approaches include the use of reflective boundary conditions and placing the first grid point off the center, which may both import calculation errors to some degree [15,18,33–35].

Let us return to the primary design of our numerical scheme. Compared with the discretization that equates the control volume with the one in purely planar flow [22,36], we base our consideration from the three-dimensional view. This contributes to maintaining the conservation property [38]. Thus, the numerical boundary condition at $r=0$ can be unified with the iterative formula of the interior cell averages. Moreover, the compatibility of the numerical boundary condition with the discretized governing equations is guaranteed. Meanwhile, mid-point interface values are incorporated into the numerical fluxes and the numerical sources to achieve a high resolution. In this paper, we focus on the GRP method to obtain the boundary value on the singular boundary $r=0$.

The GRP method, a second-order extension of the Godunov method, was originated by [1] and has been widely developed since then [2,20,31]. For the construction of numerical boundary conditions, the one-sided GRP solver has already been applied into the simulation of fluids with solid boundaries in planar flow [9,10]. Regarding the one-sided GRP for a radially symmetrical fluid, there is insufficient strict mathematical theory describing the behavior of a nonlinear wave when it approaches the center [27]. We cannot use the approach for the planar scenario to derive the corresponding solver [8]. However, with the use of acoustic approximation [5,7], we can create a prospect. The singularity of $1/r$ reminds us of the symmetrical argument $u(r=0,t)\equiv 0$. Now, it is feasible to link the limiting value of the quantity u/r with that of the space derivative $\partial u/\partial r$ using L'Hospital's rule. Compared with the extrapolation technique used in [14], our approach can enhance the spatial-temporal coupling property of the algorithm [21,39]. The wall-