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Optimal Error Estimates of the Semi-Discrete Local Discontinuous Galerkin Method and Exponential Time Differencing Schemes for the Thin Film Epitaxy Problem without Slope Selection

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Abstract. In this paper, we prove the optimal error estimates in L^2 norm of the semi-discrete local discontinuous Galerkin (LDG) method for the thin film epitaxy problem without slope selection. To relax the severe time step restriction of explicit time marching methods, we employ a class of exponential time differencing (ETD) schemes for time integration, which is based on a linear convex splitting principle. Numerical experiments of the accuracy and long time simulations are given to show the efficiency and capability of the proposed numerical schemes.

AMS subject classifications: 65M10, 35L75

Key words: Local discontinuous Galerkin method, thin film epitaxy problem, error estimates, exponential time differencing, long time simulation.

1 Introduction

In this paper, we will consider the error estimates of the local discontinuous Galerkin (LDG) method [29] for the epitaxial thin film model without slope selection [18] in 2D

$$\frac{\partial u}{\partial t} = -\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right) - \epsilon^2\Delta^2 u, \quad \boldsymbol{x}\in\Omega, \quad t\in(0,T], \tag{1.1}$$

where $\Omega=[x_0,x_0+X]\times[y_0,y_0+Y]$ is a rectangular domain, $u=u(\boldsymbol{x},t)$ is the height function, $\epsilon>0$ is a constant. This model describes the coarsening processes of the macroscopic phenomenology. The nonlinear second order term and the linear fourth order term in model (1.1) represent the Ehrlich-Schwoebel effect and the surface diffusion, respectively.

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The epitaxial thin film model without slope selection is the L^2 gradient flow with the energy functional

$$E(u)=\int_\Omega \left(F(\nabla u)+\frac{\epsilon^2}{2}|\Delta u|^2\right)d\boldsymbol{x}, \tag{1.2}$$

where $F(\nabla u)=-\frac{1}{2}\ln(1+|\nabla u|^2)$. The logarithmic term is bounded above zero but unbounded below. Some detailed discussions on this issue could be found in [23, 24] and the references cited therien. For easy presentation of the analysis, we assume a periodic boundary condition for all the variables throughout the paper. With this set of boundary conditions, model (1.1) is mass conservative and energy dissipative, i.e.,

$$\frac{d}{dt}\int_\Omega u(\boldsymbol{x},t)d\boldsymbol{x}=0, \qquad \frac{d}{dt}E(u)\le 0. \tag{1.3}$$

In [29], Xia developed a LDG method for the epitaxial thin film model without slope selection and proved the unconditional energy stability. Compared with the status of optimal error estimates for LDG methods for solving linear equations, for example, the fourth-order time-dependent problems [16] and high order wave equations [31], works on optimal L^2 error estimates for LDG methods for solving high order time-dependent nonlinear equations are few. Ji and Xu [20,21] proved optimal error estimates of the LDG method for the surface diffusion and Willmore flow of graphs. Guo, Ji and Xu [19] proved a priori error estimates in L^2 norm of the semi-discrete LDG method for the Allen-Cahn equation. In this paper, we will prove the optimal error estimates in L^2 norm of the LDG method for the thin film epitaxy problem without slope selection. The main difficulty with error estimates is the nonlinear term and the control on the auxiliary variables in the LDG method.

The discontinuous Galerkin (DG) method is a class of finite element methods using completely discontinuous piecewise polynomials as the solution and the test spaces. Reed and Hill [26] first introduced the DG method to solve steady state linear equations containing only first order spatial derivatives. Then Cockburn et al. extended the DG method for solving nonlinear equations in a series of papers [10–13]. In order to solve partial differential equations (PDEs) containing higher order spatial derivatives, the LDG method was introduced. Cockburn and Shu [14] first constructed LDG method for solving nonlinear convection diffusion equations. The idea of the LDG method is to rewrite a higher order differential equation into a system with first order equations, and then apply the DG method on the first order system. More detailed description about the LDG methods for high order time-dependent PDEs can be found in the review paper [30]. The DG and LDG methods also share several attractive properties, such as high order accuracy and flexibility on $h-p$ adaptivity and on complex geometries.

Various temporal discretization techniques have been developed for the epitaxial thin film model without slope selection, including the convex splitting schemes [2,27,28], the invariant energy quadratization (IEQ) schemes [32], the scalar auxiliary variable (SAV) [7, 8] schemes and the exponential time differencing (ETD) [3–5,8,22,25] schemes. The ETD