Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **15**, No. 3, pp. 568-582

DOI: 10.4208/aamm.OA-2021-0251
June 2023

A Priori Error Estimates for Spectral Galerkin Approximations of Integral State-Constrained Fractional Optimal Control Problems

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Received 30 July 2021; Accepted (in revised version) 24 January 2022

Abstract. The fractional optimal control problem leads to significantly increased computational complexity compared to the corresponding classical integer-order optimal control problem, due to the global properties of fractional differential operators. In this paper, we focus on an optimal control problem governed by fractional differential equations with an integral constraint on the state variable. By the proposed first-order optimality condition consisting of a Lagrange multiplier, we design a spectral Galerkin discrete scheme with weighted orthogonal Jacobi polynomials to approximate the resulting state and adjoint state equations. Furthermore, *a priori* error estimates for state, adjoint state and control variables are discussed in details. Illustrative numerical tests are given to demonstrate the validity and applicability of our proposed approximations and theoretical results.

AMS subject classifications: 65N15, 65N35, 49J20

Key words: Fractional optimal control problem, state constraint, spectral method, Jacobi polynomial, *a priori* error estimate.

1 Introduction

For the applications in mechanics, viscoelastic materials, electrical engineering and medicine, fractional order partial differential equations (FDEs, for short) are more appropriate than conventional integer-order partial differential equations to provide realistic and accurate descriptions. It is well known that, the solutions of FDEs are with naturally

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inherit weak-singularities near the boundary. How to overcome the obstacles caused by singularities during the approximations has long been an important and hot topic.

Lots of literatures are devoted to developing either efficient numerical approximations or fast algorithms for FDEs. Among these numerical methods, the spectral method is one of the efficient methods for FDEs, more details please refer to [5,9,10,15,17,25,26] and the references cited therein. To accommodate the singularity, a spectral method using weighted Jacobi polynomials has been proposed [9,29] to well deal with singularities on the boundary (endpoints in 1D). Furthermore, the corresponding numerical analyses have been rapidly explored by the development of high performance computer clusters, which also enhance the applications of spectral methods [19].

In recent years, the dynamic behaviors of fractional order optimal control problems (FOCPs, for short) have received increasing attentions of a wide-range of professionals. Naturally, following significant applications of FDEs, the theoretical and numerical discussions of FOCPs have been interesting and dominant topics. Most of the literatures focus on FOCPs, such as [2,3,11,27,30–32] for finite element methods, [16] for extended Ritz methods and [6,27] for fast algorithms. In [1], the author presented a general solution scheme for a class of FOCPs with the Riemann-Liouville fractional derivative. The authors in [18] studied a distributed FOCPs and derived the first order optimality system. The authors in [31] investigated finite element approximations of optimal control problem governed by space fractional diffusion equation with control constraints and designed a fast primal dual active set algorithm to efficiently solve the model. In [6] the authors employed finite difference schemes to approximate FOCPs with a proposed fast projected gradient algorithm.

Meanwhile, extensive results on efficient numerical algorithms for FOCPs have been proposed with spectral approximations [14,20,23,28]. Galerkin spectral approximations of FOCPs with an integral constraint on the state variable were investigated in [23]. The authors employed pseudo-spectral methods to approximate an optimal control problem governed by time-fractional diffusion equations in [13]. In [24], the authors proposed an efficient numerical scheme for solving an unconstrained convex distributed FOCPs with pseudo-spectral methods. A new fractional pseudo-spectral method for solving FOCPs was provided in [20]. Authors in [12] employed collocation methods to solve general FOCPs governed by a Caputo fractional differential equation. An optimal control problem governed by a fractional advection-diffusion-reaction equation with integral fractional Laplacian was approximated with weighted Jacobi polynomials in [22].

In this work, we focus on the advantages of spectral Galerkin methods to design efficient numerical schemes with the general Jacobi polynomials for FOCPs in Riesz derivative. And we also investigate the *a priori* error estimates for FOCPs with an integral constraint on the state variable.

The rest of this paper is organized as follows. Some preliminaries of notations, fractional calculus and Jacobi polynomials are listed in the second section. In Section 3, we introduce the FOCPs and derive the necessary first-order optimality condition from the Karush-Kuhn-Tucker theorem in continuous and discretized schemes, respectively. In