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# A New Framework of Convergence Analysis for Solving the General Nonlinear Schrödinger Equation using the Fourier Pseudo-Spectral Method in Two Dimensions

Jialing Wang[1], Tingchun Wang[1,*] and Yushun Wang[2]

[1] *School of Mathematics and Statistics, Nanjing University of Information Science & Technology, Nanjing, Jiangsu 210044, China*
[2] *Jiangsu Key Laboratory of NSLSCS, School of Mathematics Science, Nanjing Normal University, Nanjing, Jiangsu 210023, China*

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**Abstract.** This paper aims to build a new framework of convergence analysis of conservative Fourier pseudo-spectral method for the general nonlinear Schrödinger equation in two dimensions, which is not restricted that the nonlinear term is mere cubic. The new framework of convergence analysis consists of two steps. In the first step, by truncating the nonlinear term into a global Lipschitz function, an alternative numerical method is proposed and proved in a rigorous way to be convergent in the discrete $L^2$ norm; followed in the second step, the maximum bound of the numerical solution of the alternative numerical method is obtained by using a lifting technique, as implies that the two numerical methods are the same one. Under our framework of convergence analysis, with neither any restriction on the grid ratio nor any requirement of the small initial value, we establish the error estimate of the proposed conservative Fourier pseudo-spectral method, while previous work requires the certain restriction for the focusing case. The error bound is proved to be of $\mathcal{O}(h^r+\tau^2)$ with grid size $h$ and time step $\tau$. In fact, the framework can be used to prove the unconditional convergence of many other Fourier pseudo-spectral methods for solving the nonlinear Schrödinger-type equations. Numerical results are conducted to indicate the accuracy and efficiency of the proposed method, and investigate the effect of the nonlinear term and initial data on the blow-up solution.

**AMS subject classifications**: 37K, 65C, 65M, 65N

**Key words**: Framework of convergence analysis, general nonlinear Schrödinger equation, Fourier pseudo-spectral method, conservation laws, unconditional convergence, blow-up solution.

*Corresponding author.
*Email:* wangtingchun@nuist.edu.cn (T. Wang)

 

# 1 Introduction

As one of the basic equations of quantum mechanics, the nonlinear Schrödinger (NLS) equation can describe a large number of phenomena in physics, such as nonlinear optics, gravitation, quantum fluids/condensed matter physics, plasma physics, biological modeling and so forth [34,35,43,44,49]. In this paper, we consider the following general NLS equation in two-dimensional space:

$$i\partial_t u+\Delta u+f(|u|^2)u=0, \quad (x,y)\in\Omega\subseteq\mathbb{R}\times\mathbb{R}, \quad t>0, \tag{1.1}$$

with $(l_1,l_2)$–periodic boundary conditions

$$u(x+l_1,y,t)=u(x,y,t), \quad u(x,y+l_2,t)=u(x,y,t), \quad (x,y)\in\Omega, \quad t>0, \tag{1.2}$$

and initial condition

$$u(x,y,0)=\psi(x,y), \quad (x,y)\in\Omega, \tag{1.3}$$

where $i=\sqrt{-1}$ is the imaginary unit, $u=u(x,y,t)$ is the unknown complex-valued function, $\Delta:=\partial_{xx}+\partial_{yy}$ is the two-dimensional Laplacian operator. The initial condition $\psi(x,y)$ is a given $(l_1,l_2)$–periodic complex-valued function, and the given nonlinear term $f=f(s)\in C^1(\mathbb{R}^+)$ has different forms in different physical problems [19], such as the polynomial function $f(s)=s^\mu$ ($\mu>0$), the exponential function $f(s)=1-e^{-s}$, the logarithm function $f(s)=\ln(1+s)$ and the rational function $f(s)=-\frac{4s}{1+s}$. Here, we bound the computational domain $\Omega$ as $[0,l_1]\times[0,l_2]$. The NLS equation (1.1) models for the slowly varying envelop of a wave-train in conservative, dispersive, mildly nonlinear wave phenomena. It also can be obtained as the subsonic limit of the Zakharov model for Langmuir waves in plasma physics [63]. It is possible for solutions of the two-dimensional NLS equation to develop singularities at some finite time, which has been shown in [32].

Numerical contributions on mathematical and numerical aspects for the NLS equation are hot topics during the past decades. Mathematically, for the derivation, well-posedness and dynamical properties for the NLS equation, readers can refer to [8,17,41,49] and references therein. Actually, the periodic initial-boundary value problem (1.1)-(1.3) preserves many significant conservative laws. Among them, the total mass and energy conservation laws are given as [4,6,49]

$$M(t):=\int_\Omega |u(x,y,t)|^2 dxdy\equiv M(0), \qquad t\geq 0, \tag{1.4a}$$

$$E(t):=\int_\Omega \Big(|\nabla u(x,y,t)|^2-F\big(|u(x,y,t)|^2\big)\Big)dxdy\equiv E(0), \qquad t\geq 0, \tag{1.4b}$$

respectively, where

$$F(\xi)=\int_0^\xi f(s)ds, \quad |\nabla u|^2=|u_x|^2+|u_y|^2.$$