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A Maximum-Principle-Preserving Finite Volume Scheme for Diffusion Problems on Distorted Meshes

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Abstract. In this paper, we propose an approach for constructing conservative and maximum-principle-preserving finite volume schemes by using the method of undetermined coefficients, which depend nonlinearly on the linear non-conservative one-sided fluxes. In order to facilitate the derivation of expressions of these undetermined coefficients, we explicitly provide a simple constriction condition with a scaling parameter. Such constriction conditions can ensure the final schemes are exact for linear solution problems and may induce various schemes by choosing different values for the parameter. In particular, when this parameter is taken to be 0, the nonlinear terms in our scheme degenerate to a harmonic average combination of the discrete linear fluxes, which has often been used in a variety of maximum-principle-preserving finite volume schemes. Thus our method of determining the coefficients of the nonlinear terms is more general. In addition, we prove the convergence of the proposed schemes by using a compactness technique. Numerical results demonstrate that our schemes can preserve the conservation property, satisfy the discrete maximum principle, possess a second-order accuracy, be exact for linear solution problems, and be available for anisotropic problems on distorted meshes.

AMS subject classifications: 65M08, 65N08, 65N12

Key words: Finite volume scheme, maximum-principle-preserving scheme, conservative flux.

1 Introduction

Diffusion equations are used in many applications, such as radiation hydrodynamics, oil reservoir, groundwater flow simulations, etc. Within the practical requirements of these engineering applications, an accurate and reliable discretization method for the diffusion equation on distorted meshes should satisfy some fundamental properties, such as

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monotonicity, maximum principle, conservation and so on; see [31]. Among the various discretization methods for solving partial differential equations (PDEs), the finite volume method (FVM) is usually obtained by integrating the PDEs directly over the control volume. Due to its local natural conservation of fluxes, the FVM has been popular for engineering applications. A lot of literature used the FVM to solve diffusion equations on non-rectangular meshes, eg., [1, 8, 13, 14, 26]. For a more comprehensive review of the FVM for diffusion equation, we refer to [7].

As is well-known, it is crucial to preserve the non-negativity or maximum principle property of the solution to many considered problems. For example, in order to solve the thermal conduction problems, the numerical scheme must be positivity-preserving to keep the numerical simulations without violating the entropy constraints of the second law of thermodynamics, i.e., to avoid negative temperatures; see [27–29]. Also, for some practical applications, the maximum-principle-preserving numerical schemes ensure that the numerical solution is free of spurious oscillations and preserves the physical boundaries of primal problems; see [35].

In the past two or three decades, there have been many studies on constructing the positivity-preserving (sometimes called monotone) finite volume schemes for diffusion equations. In early studies, efforts were made to give specific constraints on special grids, such as Voronoi mesh, or parallelogram mesh, in order to make the numerical schemes satisfy monotonicity; see [19, 20]. When considering anisotropic problems or twisted mesh cases, more rigorous constraints need to be given in order to keep the numerical solution non-negative [3, 25]. However, The authors in [21] showed that linear monotone nine-point schemes with second-order accuracy did not exist for unconstrained general twisted quadrilateral meshes and general anisotropic diffusion. Hence, to address these difficulties, it is necessary to relax the linearity requirement of numerical schemes by considering nonlinear monotonic finite volume schemes. Recently, research on such issues has become a popular research area. Readers are referred to the literature [12, 15–17, 22, 30, 34].

However, the monotonicity in these schemes mentioned above can only maintain either the lower or upper bounds, but not both. In many applications, the maximum principle is one of the essential requirements of the discretization scheme for diffusion equations, because such a scheme can ensure the numerical solution is free of spurious oscillations and preserves the physical boundaries of the diffusion problem. In [23], the authors proposed a finite volume scheme for highly anisotropic diffusion operators on triangular meshes. This scheme satisfies a discrete version of the classical local maximum (and minimum) principle for elliptic equations without geometrical constraints on the mesh and constraining conditions on the anisotropy ratio. This method was then extended to very general grids in any dimension of space in [6]. In order to obtain an interpolation-free scheme, the authors of [11, 18] used the edge unknowns in the harmonic averaging expressions. In addition to these schemes, there is a large amount of literature devoted to the construction of cell-centered finite volume schemes satisfying the maximum principle, e.g., [4, 5, 31, 35–37]. In the above schemes, the unknowns at vertices, or midpoints