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The Convergence of Euler-Maruyama Method of Nonlinear Variable-Order Fractional Stochastic Differential Equations

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Abstract. In this paper, we first prove the existence and uniqueness theorem of the solution of nonlinear variable-order fractional stochastic differential equations (VFS-DEs). We futher constructe the Euler-Maruyama method to solve the equations and prove the convergence in mean and the strong convergence of the method. In particular, when the fractional order is no longer varying, the conclusions obtained are consistent with the relevant conclusions in the existing literature. Finally, the numerical experiments at the end of the article verify the correctness of the theoretical results obtained.

AMS subject classifications: 34A12, 65C30, 90B36

Key words: Variable-order Caputo fractional derivative, Stochastic differential equations, Euler-Maruyama method, convergence, multiplicative noise.

1 Introduction

In recent years, fractional differential equations have received more and more attention due to their applications in various disciplines such as Mechanics, Physics, Electrical engineering and Control theory [1–3]. However, some recent developments in fractional calculus indicate that a large number of natural phenomena can be modeled by variable fractional differential equations. The main advantage of variable-order fractional derivatives is that they exhibit memory effects that vary with space and time positions [4, 5]. Therefore, the research on variable fractional differential equations has attracted widespread attention and there have been some research results. In [6, 7], the basic properties of variable-order fractional integrators and differentiators are studied.

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In [8–11], several numerical methods, including matrix approach, reproducing kernel method and finite difference method for solving variable-order fractional differential equations were proposed. Jia and Xu et al. in [12] concerned with an efficient numerical scheme for variable order fractional functional boundary value problems. The algorithm relies on the simplified reproducing kernel method. Zhang and Liu et al. proposed an implicit Euler approximation for a new space-time variable fractional order advection-dispersion equation and investigate the stability and convergence of the approximation in [13]. Moghadam and Arabameri et al. in [14] alternatively solved that equation using Radial basis functions and Jacobi spectral collocation methods, which are judged, and the results are far better than [13]. Wang and Zheng proved the wellposedness of a nonlinear variable-order fractional differential equation and the regularity of its solution in [15]. Soon after, they proved the wellposedness of a variable-order linear time-fractional diffusion partial differential equation in multiple space dimensions in [16].

There are also more articles on stochastic differential equations(SDEs) involving fractional derivatives. For example, Anh and Mcvinish proposed a fractional differential equation with Lévy in [17]. Jumarie proposed nonlinear stochastic fractional differential equations (SFDEs) in [18]. Subsequently, Sakthivel and Revathiet et al. in [19–21] discussed the existence and uniqueness of solutions of nonlinear SFDEs in Hilbert space. Wang and Sun studied the weak convergence of the Eluer method for a class of SFDEs driven by an additive noise in [22]. Yan and Xiao proved the strong convergence of the Euler method for a class of SFDEs driven by a multiplicative noise in [23]. Doan constructd Euler-Maruyama and exponential Euler-Maruyama method for Caputo SFDEs and show the strong convergence rate of the two methods in [24]. Zhang and Wang et al. gave the results of the existence, uniqueness, stability and Hölder regularity of the solution of a class of SFDEs under certain conditions and the convergence and stability of Euler method were analyzed. Moreover the application of the model and method in Newton-Leipnik system and financial chaos system were given in [25]. Yang and Zhao in [26] used finite element methods in the physical space domain and the Euler method in the time domain to propose a spatial finite element semi-discrete scheme and a spatio-temporal full discrete scheme for solving a class of nonlinear backward stochastic partial differential equations. Zhou and Sun developed an explicit third order onestep method for solving decoupled forward backward SDEs in [27].

However, there are few studies that combine variable fractional derivatives with SDEs. In this paper, we focus on the convergence analysis of the Euler-Maruyama method for nonlinear variable-order fractional stochastic differential equations (VFSDEs).

Let $(\Omega,\mathcal{F},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right-continuous and each $\mathcal{F}_t, t\geq 0$, contains all P-null sets in $\mathcal{F}$. Let $|\cdot|$ denotes the Euclidean norm in R^d and the trace norm in $R^{d\times r}$. We define

$$a\vee b:=\max\{a,b\}, \qquad a\wedge b:=\min\{a,b\}.$$