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A Partial RKDG Method for Solving the 2D Ideal MHD Equations Written in Semi-Lagrangian Formulation on Moving Meshes with Exactly Divergence-Free Magnetic Field

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Abstract. A partial Runge-Kutta Discontinuous Galerkin (RKDG) method which preserves the exactly divergence-free property of the magnetic field is proposed in this paper to solve the two-dimensional ideal compressible magnetohydrodynamics (MHD) equations written in semi-Lagrangian formulation on moving quadrilateral meshes. In this method, the fluid part of the ideal MHD equations along with z-component of the magnetic induction equation is discretized by the RKDG method as our previous paper [47]. The numerical magnetic field in the remaining two directions (i.e., x and y) are constructed by using the magnetic flux-freezing principle which is the integral form of the magnetic induction equation of the ideal MHD. Since the divergence of the magnetic field in 2D is independent of its z-direction component, an exactly divergence-free numerical magnetic field can be obtained by this treatment. We propose a new nodal solver to improve the calculation accuracy of velocities of the moving meshes. A limiter is presented for the numerical solution of the fluid part of the MHD equations and it can avoid calculating the complex eigen-system of the MHD equations. Some numerical examples are presented to demonstrate the accuracy, non-oscillatory property and preservation of the exactly divergence-free property of our method.

AMS subject classifications: 76M10, 76N15, 76W05

Key words: Ideal compressible MHD equations, semi-Lagrangian formulation, exactly divergence-free magnetic field, Runge-Kutta discontinuous Galerkin method.

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1 Introduction

In this paper, we will continue to focus on the RKDG discretization proposed in [47] for solving the 2D ideal MHD equations written in semi-Lagrangian formulation

$$\begin{cases} \dfrac{d}{dt}(J\mathbf{U})+J\nabla\cdot(\mathbf{F},\mathbf{G})=0, & \text{(1.1a)}\\ \dfrac{d}{dt}(J\mathbf{B})+J\nabla\cdot(\mathbf{F}^B,\mathbf{G}^B)=0, & \text{(1.1b)}\end{cases}$$

with

$$\begin{aligned}
&\mathbf{U}=(\rho,\rho u_x,\rho u_y,\rho u_z,B_z,\varepsilon)^T, \qquad \mathbf{B}=(B_x,B_y)^T,\\
&\mathbf{F}=(0,P_{tot}-B_x^2,-B_xB_y,-B_xB_z,-u_zB_x,P_{tot}u_x-B_x(\mathbf{v}\cdot\mathbb{B}))^T,\\
&\mathbf{G}=(0,-B_xB_y,P_{tot}-B_y^2,-B_yB_z,-u_zB_y,P_{tot}u_y-B_y(\mathbf{v}\cdot\mathbb{B}))^T,\\
&\mathbf{F}^B=(-u_xB_x,-u_yB_x)^T, \qquad \mathbf{G}^B=(-u_xB_y,-u_yB_y)^T,
\end{aligned}$$

where J denotes the Jacobian of the mapping from a fluid particle's initial coordinates (X,Y) (Lagrangian coordinates) to its coordinates (x,y) (Eulerian coordinates) at time $t>0$ and can be written as

$$J(X,Y,t)=\begin{vmatrix} \dfrac{\partial x}{\partial X} & \dfrac{\partial y}{\partial X}\\ \dfrac{\partial x}{\partial Y} & \dfrac{\partial y}{\partial Y}\end{vmatrix}, \tag{1.2}$$

ρ, $\mathbf{v}=(u_x,u_y,u_z)^T$ and $\mathbb{B}=(B_x,B_y,B_z)^T$ are the density, the fluid velocity and the magnetic field respectively, $P_{tot}=P+P_{mag}$ is the total pressure which is made up of the thermal pressure P and the magnetic pressure $P_{mag}=\frac{1}{2}|\mathbb{B}|^2$, and ε denotes the total energy which consists of internal, kinetic, and magnetic energies,

$$\varepsilon=\frac{P}{\gamma-1}+\frac{1}{2}\rho|\mathbf{v}|^2+\frac{1}{2}|\mathbb{B}|^2,$$

with $\frac{P}{\gamma-1}$ for the internal energy and γ for the ratio of the specific heats. For the simplicity of expression, the magnetic permeability is set to be unity in this paper.

It is worth noting that (1.1a) is the semi-Lagrangian for the fluid field part of the ideal MHD equations and the z-component of the magnetic induction equations while (1.1b) is the semi-Lagrangian for the (x,y)-components of the magnetic induction equation.

According to the physical principle that there are no magnetic monopoles, this system is complemented with the following divergence-free constraint $\nabla\cdot\mathbb{B}=0$ on the magnetic field. In 2D case, this constraint turns into

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}=0. \tag{1.3}$$