Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **15**, No. 5, pp. 1142-1165

DOI: 10.4208/aamm.OA-2022-0200
October 2023

A Finite Element Variational Multiscale Method for Stationary Incompressible Magnetohydrodynamics Equations

Huayi Huang, Yunqing Huang and Qili Tang*

Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Key Laboratory of Intelligent, Computing & Information Processing of Ministry of Education, School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, China

Received 24 July 2022; Accepted (in revised version) 30 September 2022

Abstract. In this paper, we propose a variational multiscale method (VMM) for the stationary incompressible magnetohydrodynamics equations. This method is defined by large-scale spaces for the velocity field and the magnetic field, which aims to solve flows at high Reynolds numbers. We provide a new VMM formulation and prove its stability and convergence. Finally, some numerical experiments are presented to indicate the optimal convergence of our method.

AMS subject classifications: 65N30, 65N12, 65M60

Key words: Variational multiscale method, stationary incompressible magnetohydrodynamics, large-scale spaces, stability and convergence, high Reynolds numbers.

---

1 Introduction

In this paper, we consider the stationary incompressible magnetohydrodynamics (MHD) problem in Ω:

$$-R_e^{-1}\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-S_c\mathrm{curl}\mathbf{b}\times\mathbf{b}=\mathbf{f}, \qquad \mathrm{div}\mathbf{u}=0, \tag{1.1a}$$

$$R_m^{-1}S_c\mathrm{curl}(\mathrm{curl}\mathbf{b})-S_c\mathrm{curl}(\mathbf{u}\times\mathbf{b})-\nabla r=\mathbf{g}, \qquad \mathrm{div}\mathbf{b}=0, \tag{1.1b}$$

subject to the boundary conditions

$$\mathbf{u}|_{\partial\Omega}=0, \quad \mathbf{b}\cdot\mathbf{n}|_{\partial\Omega}=0, \quad \mathbf{n}\times\mathrm{curl}\mathbf{b}|_{\partial\Omega}=0, \quad r|_{\partial\Omega}=0. \tag{1.2}$$

*Corresponding author.
Emails: huanghy@smail.xtu.edu.cn (H. Huang), huangyq@xtu.edu.cn (Y. Huang), tangqili@xtu.edu.cn (Q. Tang)

Here, Ω is a bounded Lipschitz domain in R^d ($d=2$ or 3), $\mathbf{u}$ the velocity field, $\mathbf{b}$ the magnetic field, p the pressure, r a Lagrange multiplier, and $\mathbf{f}\in H^{-1}(\Omega)^d$ and $\mathbf{g}\in L^2(\Omega)^d$ given source terms. $\mathbf{n}$ respects the outward normal unit vector on $\partial\Omega$, and R_e, R_m and S_c are the hydrodynamic Reynolds number, magnetic Reynolds number and coupling number, respectively.

Furthermore, by taking the divergence of the first equation of Eq. (1.1b), we obtain $-\Delta r=\mathrm{div}\mathbf{g}$ in Ω. Since $\mathbf{g}$ is divergence-free in practical application, combining with the fourth boundary condition of Eq. (1.2) gives $r\equiv 0$. This means that the adding Lagrange multiplier is reasonable.

MHD equations show the coupling of hydrodynamics phenomena and electromagnetic phenomena. In recent years, a large number of applications of numerical method to MHD flows reveal for its popularity [1–9]. In [1], a mixed finite element approach for MHD system was proposed to solve a double saddle point problem by constructing a Lagrange multiplier. The adding Lagrange multiplier allows us to solve the MHD system in a non-convex domain. In [6], div-conforming Brezzi-Douglas-Marini elements were used for the velocity field and first family of curl-conforming Nédélec elements were employed for the magnetic field. In order to circumvent LBB condition, some stabilized finite element methods were introduced in [2]. Furthermore, for dealing with MHD nonlinear terms, the study on iterative methods has a lot of contributions in [10–15].

Hughes et al. [16] provided a variational framework for subgrid-scale modeling, called variational multiscale method. It is shown that the flow can be decomposed into resolved scales and unresolved scales. In the beginning, the method was performed for the large eddy simulation by the scale separation. Collis [17] extended this method to solve the incompressible Navier-Stokes equations by using a three-level partition, which defined the large-scale, small-scale (resolved scales) and unresolved scales. This method was successfully used to computational fluid dynamics at high Reynolds number, see [18–26] and the references therein. Here, John et al. [18–20] developed a finite element variational multiscale method for incompressible flow, by defining a large-scale space L^H for the gradient of the velocity field. Zheng et al. [21, 24] provided some variational multiscale methods for incompressible flow based on two local Gauss integrations. In addition, residual-free bubbles (RFB) method [27–30] based on subgrid scales was introduced in a different way. It requires those unresolvable parts vanishing on the boundaries of all elements. Thus it gives the boundary value problem on every element, namely, solving these subgrid-scale models obtains residual-free bubbles. Then we can compute the effect of unresolved scales onto resolved scales by these bubble functions.

In this paper, the VMM by defining some large-scale spaces for the velocity field and the magnetic field is considered to solve the MHD system (1.1a)-(1.2). In this approach, the unresolvable part is projected into some proper spaces and added to the original discretization formulation. It implies that the method simulates the effect of unresolved scales onto resolved scales. We use H^1-conforming elements for the velocity field and curl-conforming Nédélec elements [31, 32] for the magnetic field. Then we consider a multiscale formulation of MHD system, by adding some artificial terms.