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Implicit Runge-Kutta-Nyström Methods with Lagrange Interpolation for Nonlinear Second-Order IVPs with Time-Variable Delay

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Abstract. This paper deals with nonlinear second-order initial value problems with time-variable delay. For solving this kind of problems, a class of implicit Runge-Kutta-Nyström (IRKN) methods with Lagrange interpolation are suggested. Under the suitable condition, it is proved that an IRKN method is globally stable and has the computational accuracy $\mathcal{O}(h^{\min\{p,\mu+\nu+1\}})$, where p is the consistency order of the method and $\mu,\nu \geq 0$ are the interpolation parameters. Combining a fourth-order compact difference scheme with IRKN methods, an initial-boundary value problem of nonlinear delay wave equations is solved. The presented experiments further confirm the computational effectiveness of the methods and the theoretical results derived in previous.

AMS subject classifications: 65L03, 65L04, 65L80

Key words: Nonlinear second-order initial value problems, time-variable delay, Lagrange interpolation, implicit Runge-Kutta-Nyström methods, error analysis, global stability.

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1 Introduction

The initial value problems (IVPs) of second-order delay differential equations (DDEs) are a kind of important models for describing the practical scientific phenomena arising in vibration mechanics, mechanical engineering, biodynamics, automatic control and the other related fields (see e.g., [1, 2]). Nevertheless, for the IVPs of nonlinear second-order DDEs, it is very difficult to obtain their exact solutions. Hence, in the recent years, ones have begun to develop various numerical methods to solve this kind of problems. For

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example, Papageorgiou & Famelis [3] constructed explicit Runge-Kutta-Nyström methods, Martín & García [4] presented variable-stepsize multistep methods, Ramadan, El-Sherbeiny & Sherif [5] proposed polynomial spline methods, Seong & Majid [6] gave an Adams-Moulton-type method, Kherd, Omar, Saaban & Adeyeye [7] suggested operational matrix methods, Zhang& Wang [8] derived generalized Strömer-Cowell methods and Li & Zhou [9] further adapted the methods in [8] into block generalized Strömer-Cowell methods.

In the existing numerical methods for solving IVPs of second-order DDEs, one-step methods usually have higher computational efficiency than multistep methods with the same consistency order since the computational procedure of an one-step method is self-starting. In view of this, in the actual computation, ones often prefer to use one-step methods to solve the problems. As an example, the mentioned-above explicit Runge-Kutta-Nyström methods are just a type of one-step methods. It is well-known that explicit methods do not work for stiff problems as the boundedness of their stability regions confines the computational stepsize into excessively small and thus leads to an unsuccessful computation. In order to overcome this defect, in the present paper, we will consider implicit Runge-Kutta-Nyström methods with Lagrange interpolation to solve nonlinear second-order IVPs with time-variable delay.

The rest of this paper is organized as follows. In Section 2, by adapting the standard IRKN (see e.g., [10–12]) methods and combining the Lagrange interpolation, we construct a class of new IRKN methods to solve nonlinear second-order IVPs with time-variable delay. In Section 3, we perform an error analysis for IRKN methods and proved that the methods can arrive at the computational accuracy $\mathcal{O}(h^{\min\{p,\mu+\nu+1\}})$ under the suitable condition, where p is the consistency order of the method and $\mu,\nu\geq 0$ are the interpolation parameters. In Section 4, we study nonlinear global stability of IRKN methods and derive a global stability criterion of the methods. In Section 5, with a combination of the fourth-order compact difference scheme and IRKN methods, we present an application to an initial-boundary value problem (IBVP) of nonlinear delay wave equations. The presented numerical experiments further verify the computational effectiveness of the methods and the theoretical results obtained in previous sections.

2 Nonlinear second-order IVPs with time-variable delay and their IRKN methods

Consider the following nonlinear d-dimensional second-order IVPs with time-variable delay $\tau(t)>0$:

$$y''(t)=f(t,y(t),y(t-\tau(t))), \qquad t\in[t_0,T]; \tag{2.1a}$$

$$y(t)=\varphi(t), \quad y'(t)=\varphi'(t), \qquad t\in[\tau_0,t_0], \tag{2.1b}$$