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New Finite Volume Mapped Unequal-Sized WENO Scheme for Hyperbolic Conservation Laws

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Abstract. This article designs a new fifth-order finite volume mapped unequal-sized weighted essentially non-oscillatory scheme (MUS-WENO) for solving hyperbolic conservation laws on structured meshes. One advantage is that the new mapped WENO-type spatial reconstruction is a convex combination of a quartic polynomial with two linear polynomials defined on unequal-sized central or biased spatial stencils. Then we propose the new mapped nonlinear weights and new mapping function to decrease the difference between the linear weights and nonlinear weights. This method has the characteristics of small truncation errors and high-order accuracy. And it could give optimal fifth-order convergence with a very tiny ε even near critical points in smooth regions while suppressing spurious oscillations near strong discontinuities. Compared with the classical finite volume WENO schemes and mapped WENO (MWENO) schemes, the linear weights can be any positive numbers on the condition that their summation is one, which greatly reduces the calculation cost. Finally, we propose a new modified positivity-preserving method for solving some low density, low pressure, or low energy problems. Extensive numerical examples including some unsteady-state problems, steady-state problems, and extreme problems are used to testify to the efficiency of this new finite volume MUS-WENO scheme.

AMS subject classifications: 65M60, 35L65

Key words: Mapped WENO scheme, finite volume, unequal-sized stencil, mapping function, steady-state problem, extreme problem.

1 Introduction

In this article, a new fifth-order finite volume mapped unequal-sized weighted essentially non-oscillatory (MUS-WENO) scheme is designed for hyperbolic conservation laws.

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We first introduce some features of this finite volume MUS-WENO scheme. Different from the classic finite volume high-order WENO scheme [35], a new space reconstruction methodology is used to construct one quartic polynomial and two linear polynomials on unequal-sized spatial stencils. It effectively avoids contact discontinuities or strong shocks that lie in all equal-sized central or biased spatial stencils [35]. And the linear weights can be any positive number that summation to 1, which significantly saves the calculation cost. Based on [23], we design a new mapping function to decrease the difference between the linear weights and nonlinear weights. It could achieve optimal fifth-order accuracy with very tiny ε even near the critical points in smooth regions while having sharp shock transitions in discontinuous regions. This new finite volume MUS-WENO scheme has small numerical errors in L^1 and L^∞ norms inside smooth regions. Finally, this finite volume MUS-WENO scheme with a very tiny ε can compute some unsteady-state problems without losing the designed order of accuracy at critical points in smooth regions, some steady-state problems without introducing big average residue, and some extreme problems containing low density, low pressure, or low energy, respectively.

So far, many numerical schemes were studied for solve compressible hyperbolic conservation laws with various fluid structures. As early as 1984, Colella et al. [6] innovatively designed the piecewise parabolic method that used a four-point central spatial stencil to represent the interfacial values and the values were applied to suppress nonphysical oscillations at discontinuities. In 1991, Leonard [29] proposed the ULTIMATE conservative difference scheme for the first time and effectively solved one-dimensional unsteady advection problems. The results show that the limiting methods could reduce the accuracy from theoretical optimal order to first-order accuracy even at critical points in smooth regions. The idea of the PMM method can be traced back to the MUSCL scheme [39] and Godunov's scheme [15]. The construction of high-order schemes has always been the focus of research. Harten and Osher [22] proposed a new TVD scheme [19] and designed the new ENO schemes. Harten et al. [21] applied the new ENO schemes to simulate 1D hyperbolic conservation laws problems. In the same year, Harten [20] innovatively designed a 2D extension of the ENO schemes. In 1992, Casper [4], and Casper and Atkins [5] presented the ENO schemes for solving hyperbolic conservation laws. In 1994, Liu et al. [33] first designed a third-order finite volume weighted ENO (WENO) scheme based on the ENO scheme. In 1996, Jiang et al. [27] proposed a fifth-order finite difference WENO scheme and extended it to multi-dimensional cases. In 1999, Hu et al. [26] designed the higher-order WENO schemes on unstructured meshes. In 2004, Titarev and Toro [38] designed the finite-volume WENO schemes for solving 3D hyperbolic conservation laws. In 2009, Zhang et al. [46] designed the third-order WENO scheme on the tetrahedral meshes. Due to the construction process of the traditional high-order WENO schemes, the linear weight may have negative values. So, special handling of negative linear weights was required, which increased the computational cost. In 2017, Zhu and Qiu [49] overcame this difficulty.

In 2005, Henrick et al. [23] designed a new mapping function to solve the difficulty