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Numerical Analysis of Stabilized Second Order Semi-Implicit Finite Element Methods for the Phase-Field Equations

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Abstract. In this paper, we consider two stabilized second-order semi-implicit finite element methods for solving the Allen-Cahn and Cahn-Hilliard equations. Stabilized semi-implicit schemes are used for temporal discretization, and the finite element method is used for spatial discretization. It is shown that by adding a single linear term that is of the same order with the truncation error in time, the proposed methods are all unconditionally energy stable. Error estimates for the two schemes are also established. Numerical examples are presented to confirm the accuracy, efficiency and stability of the proposed methods.

AMS subject classifications: 65N15, 65N30, 65M15

Key words: Allen-Cahn equation, Cahn-Hilliard equation, stabilized semi-implicit method, energy stable, error estimation.

1 Introduction

We consider in this work the numerical approximation of the Allen-Cahn equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \epsilon \Delta u + F'(u) = 0, & (x,t) \in \Omega \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \end{cases} \tag{1.1}$$

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and the Cahn-Hilliard equation

$$\begin{cases} \dfrac{\partial u}{\partial t}+\Delta(\epsilon\Delta u-F'(u))=0, & (x,t)\in\Omega\times(0,T], \\ u(x,0)=u_0(x), & x\in\Omega, \\ \dfrac{\partial u}{\partial n}=\dfrac{\partial(\epsilon\Delta u-F'(u))}{\partial n}=0, & (x,t)\in\partial\Omega\times(0,T]. \end{cases} \tag{1.2}$$

Here $\Omega\subset R^d$ $d=(1,2,3)$ is a bounded domain, $\partial\Omega$ denotes the Lipschitz boundary of Ω. n denotes the unit outward normal vector of $\partial\Omega$. $T>0$ is a fixed constant. The parameter ϵ which models the effect of interfacial energy is small but always larger than zero. The Ginzburg-Landau double well potential $F(u)=(u^2-1)^2/4$ is considered, and the function $u(x,t)$ is a distribution function of the concentration for one of the two metallic components of the alloy. As we all know that the Allen-Cahn and Cahn-Hilliard equation can be regarded as the gradient flow of the following Liapunov energy functional

$$E(u)=\int_\Omega\Big(\frac{\epsilon}{2}|\nabla u|^2+F(u)\Big)dx \tag{1.3}$$

in L^2-space and H^{-1}-space, respectively. If we take the inner product for the first equation in (1.1) with u_t, we can obtain the following equality

$$(u_t,u_t)+\epsilon(\nabla u,\nabla u_t)+(u^3,u_t)-(u,u_t)=0,$$

and it is easy to show that for the the Allen-Cahn equation (1.1),

$$\frac{dE(u(t))}{dt}=\int_\Omega\epsilon\nabla u\nabla u_t+(u^3-u)u_t dx=-(u_t,u_t)\le 0. \tag{1.4}$$

Similarly, if we take the inner product for the first equation in (1.2) with $-\Delta^{-1}u_t$, we have

$$\|u_t\|_{-1}^2-(\nabla\mu,\nabla\mu)=0.$$

Here $\mu=\epsilon\Delta u-F'(u)$, and the H^{-1} norm $\|\cdot\|_{-1}$ is defined in the next section. Hence, we could prove that for the Cahn-Hilliard equation (1.2),

$$\frac{dE(u(t))}{dt}=\int_\Omega\epsilon\nabla u\nabla u_t+(u^3-u)u_t dx=-(\nabla\mu,\nabla\mu)=-\|u_t\|_{-1}^2\le 0. \tag{1.5}$$

Eqs. (1.4)-(1.5) indicate that the Allen-Cahn and Cahn-Hilliard equation possess energy-decay property: the total energy is decreasing in time, and the following energy law holds:

$$E(u(t_2))\le E(u(t_1)),\quad \forall t_1<t_2\in(0,T]. \tag{1.6}$$

As two famous phase-field models, the Allen-Cahn equation originated from the work by Allen and Cahn [1] in which a diffusive interfacial model is built to describe the phenomenon of antiphase domain coarsening in a binary alloy. While the Cahn-Hilliard