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The Nehari Manifold for a Class of Singular ψ-Riemann-Liouville Fractional with p-Laplacian Operator Differential Equations

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Abstract. Using Nehari manifold method combined with fibring maps, we show the existence of nontrivial, weak, positive solutions of the nonlinear ψ-Riemann-Liouville fractional boundary value problem involving the p-Laplacian operator, given by

$$(P)\quad \begin{cases} -{}_tD_T^{\alpha,\psi}\Big(|{}_0D_t^{\alpha,\psi}(u(t))|_0^{p-2}D_t^{\alpha,\psi}(u(t))\Big)=\dfrac{\lambda g(t)}{u^{\gamma}(t)}+f(t,u(t)), \quad t\in(0,T),\\ u(0)=u(T)=0, \end{cases}$$

where $\lambda>0$, $0<\gamma<1<p$ and $\frac{1}{p}<\alpha\leq 1$, $g\in C([0,T])$ and $f\in C^1([0,T]\times\mathbb{R},\mathbb{R})$. A useful examples are presented in order to illustrate the validity of our main results.

AMS subject classifications: 26A33, 34A08, 34B15, 35J20

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1 Introduction

Comparing different types of differential equations, we see that in the last years, the fractional calculus take a wide area in researches due to its several advantages and applications in problems contain infinity and hereditary memory, we refer the reader to [5, 11, 15, 20]. The fractional calculus tracer notes the appearance of another type of fractional derivative besides the classical and fractional-order differential and integral operators,

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which is called the ψ-fractional derivative as an instance, Riemann-Liouville [13], Caputo [17], Hilfer [21], etc. Nehari manifold method is a very helpful procedure in the calculus of variations presented in the early 1960's. In fact, Nehari method is used to show that a boundary value problem for certain second order nonlinear ordinary differential equations in an open interval (a,b) has a non-trivial solution that can be obtained through minimization problems. On the other hand, elliptical differential problems [4,6,12,16] can be solved by fibering maps method.

In this works, we study the following nonlinear ψ-Riemann-Liouville fractional problem with involving $p-$Laplacian operator,

$$(P)\quad \begin{cases} -{}_tD_T^{\alpha,\psi}\Big(|{}_0D_t^{\alpha,\psi}(u(t))|_0^{p-2}D_t^{\alpha,\psi}(u(t))\Big)=\dfrac{\lambda g(t)}{u^{\gamma}(t)}+f(t,u(t)), \quad t\in(0,T),\\ u(0)=u(T)=0, \end{cases}$$

where ${}_\cdot D_\cdot^{\alpha,\psi}$ is the ψ-Riemann-Liouville fractional derivative of order α, $\lambda>0$, $0<\gamma<1<p<r$, $\frac{1}{p}<\alpha\le 1$, $g\in C([0,T])$ and $f\in C^1([0,T]\times\mathbb{R},\mathbb{R})$.

In the literature, nonlinear one-term FDEs of the form

$$-{}_tD_T^{\alpha}\Big(|{}_0D_t^{\alpha}(u(t))|_0^{p-2}D_t^{\alpha,\psi}(u(t))\Big)=\lambda g(t)|u^{q-2}(t)|+f(t,u(t))$$

have been considered by many authors, see examples [19], where ${}_\cdot D_\cdot^{\alpha}$ is the Riemann-Liouville fractional derivative, $\lambda>0$, $2<r<p<q$, $\frac{1}{2}<\alpha\le 1$, $g\in C([0,1])$ and $f\in C^1([0,1]\times\mathbb{R},\mathbb{R})$, and [10], where ${}_\cdot D_\cdot^{\alpha}$ is the Riemann-Liouville fractional derivative, $\lambda>0$, $1<r<p<q$, $\frac{1}{p}<\alpha\le 1$, $g\in C([0,1])$ and $f(t,u)=\nabla W(t,u)$ is the gradient of $W(t,u)$ at u and $W\in C^1([0,1]\times\mathbb{R}^n,\mathbb{R})$ is homogeneous of degree r.

This paper is organized as follows, in Section 2 we present some preliminaries and usefully results which will be used in the following section. Section 3 is devoted to the proof of necessary lemmas needed to prove the main results. In Section 4, we use the Nehari Manifold to prove the main Theorem which give the hypothesis to have a two nontrivial solutions of (P). Finally, in Section 5, we present an important examples in order to illustrate the main results of this article.

2 Preliminaries

In this section, we recall some background definitions and results on the theory of ψ-fractional calculus, in particular the ψ-Riemann-Liouville fractional derivative. Through this paper, $-\infty\le a<b\le\infty$ and $\psi\in C^1((a,b))$ an increasing positive function such that $\psi'(x)\neq 0$, for all $x\in I$.

Definition 2.1 ([14,18]). *Let h be an integrable function defined on $[a,b]$. For all $x\in[a,b]$, the left and right, respectively, fractional integrals of a function h with respect to another function ψ*